Stability of flat galaxies

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Roman Fiřt

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Contents

Abstract

In this thesis we investigate the existence and properties of stationary solutions of the flat Vlasov-Poisson system. This system of partial differential equations can be used as a model of extremely flat astronomical objects and is a combination between the two-dimensional motion of particles and the three-dimensional interaction through their gravitational potential.

The steady-states are constructed by the so-called energy-Casimir method developed by Guo and Rein, where the minimization of a suitable energy functional provides existence and non-linear stability of such steady-states. This thesis proceeds as follows. In Chapter 3 we adapt the reduction procedure for the energy-Casimir functional known in the full three-dimensional case to get an existence and stability for a large group of polytropic stationary solutions against all planar perturbations (not necessarily axially symmetric). We also describe the connection between stability for the flat Vlasov-Poisson system and stability for the flat Euler-Poisson system, a system describing dynamics of a thin disk of ideal non-viscous fluid.

Chapter 4 investigates the "limit" polytropic steady-state, the Kuzmin disk. The Kuzmin disk is widely used in the astrophysical literature as a model for various flat astronomical objects. Its limiting properties can be understood in the sense, that it has finite mass, but the support is unbounded (as opposed to all polytropes with lower polytropic index, which all have compact support). We prove also its non-linear stability against general planar perturbations.

In Chapter 5 we introduce the new model describing a flat galaxy inside a halo of dark matter. We show some a-priori estimates of the total energy and prove properties of the energy minimizer.

In dieser Dissertation beschäftige ich mich mit dem flachen Vlasov-Poisson-System, einem System partieller Differentialgleichungen, welches als Modell für hinreichend flache astronomische Objekte weithin in Gebrauch ist. Im Zentrum der Arbeit steht die Entwicklung einer Existenztheorie für speziell geartete stationäre Lösungen dieses Systems und die Untersuchung deren weiterer Eigenschaften. Die wesentliche Schwierigkeit bei der flachen Variante des Vlasov-Poisson-Systems besteht in der Kopplung von zweidimensionaler Teilchenbewegung und "dreidimensionaler Wechselwirkung" durch gravitative Kräfte.

Die Konstruktion der stationären Zustände erfolgt mit Hilfe der sogenannten Energie-Casimir-Methode, welche von GUO und REIN entwickelt wurde. Hierbei kann sowohl die Existenz als auch die nichtlineare Stabilität von stationären Zuständen aus einem Minimierungsprinzip gewonnen werden.

Diese Arbeit gliedert sich folgendermaßen: In Kapitel 3 wird die Reduktionsmethode für die Energie-Casimir-Funktionale modifiziert, um die Existenz und die Stabilität für eine große Klasse sogenannter Isotrope, das sind Lösungen, deren Verteilungsfunktion nur von der Teilchenenergie abhängt, gegen allgemeine flache Störungen (nicht nur axialsymmetrische) zu beweisen. Hier wird auch der Zusammenhang zwischen der Stabilität des Vlasov-Poisson Systems und der Stabilität des Euler-Poisson Systems besprochen. Das letztgenannte System wird als Modell für eine ideale nichtviskose Flüssigkeit benutzt.

In Kapitel 4 untersuche ich den unter dem Namen Kuzmin Disk bekannten Grenzfall der polytropen Lösungen. Dieser ist ein in der Astrophysik weithin akzeptiertes Modell für flache Objekte. Die Besonderheit dieser Grenzlösung besteht darin, dass dieser Zustand zwar endliche Masse hat, aber sein Träger - im Unterschied zu allen anderen Polytropen - unbeschränkt ist. Ich beweise in dieser Arbeit die nichtlineare Stabilität des Kuzmin Disks gegen allgemeine flache Störungen.

In Kapitel 5 führe ich ein neues Modell ein, mit dem die Dynamik einer Galaxie umgeben von einer Halo aus dunkler Materie beschrieben wird. Ich beweise die a-priori Abschätzungen der Energie und beschäftige mich eingehend mit den Eigenschaften des Minimizers.

Notation

$\mathbb{R}^n$	n-dimensional Euclidean space with coordinates $(x_1, \ldots, x_n)$,
$\mathbb{R}^+$	$\{x \in \mathbb{R} : x > 0\}$,
$\mathbb{R}_0^+$	$\{x \in \mathbb{R} : x \geq 0\}$,
$C^k(\mathbb{R}^n)$	space of all k times continuously differentiable functions,
$C_{\mathrm{c}}^k(\mathbb{R}^n)$	space of all functions in C^k with compact support,
B_R	$\{x \in \mathbb{R}^n : \lvert x\rvert \leq R\}$,
B_{R_1,R_2}	$\{x \in \mathbb{R}^n : R_1 \leq \lvert x\rvert \leq R_2\}$,
$\mathbf{1}_M$	indicator function of a set M,
$L^p(\mathbb{R}^n)$	Lebesgue space,
$\lvert\lvert \cdot \rvert\rvert_{L^p(\mathbb{R}^n)}$, $\lvert\lvert \cdot \rvert\rvert_p$	Lebesgue norm, $$\lvert\lvert f\rvert\rvert_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \lvert f(x)\rvert^p\right)^{1/p},$$
$L_+^p(\mathbb{R}^n)$	set of all function from $L^p(\mathbb{R}^n)$, which are non-negative almost everywhere,
$L_{\mathrm{w}}^p(\mathbb{R}^n)$	weak Lebesgue space – space of all measurable functions f such that $$\sup_{\alpha>0} \alpha \lvert\{x : \lvert f(x)\rvert > \alpha\}\rvert^{1/p} < \infty,$$
$(\cdot)_+$	positive part of a function, $$(f(x))_+ := \max\{0, f(x)\},$$
$\frac{\partial f}{\partial x}$	partial derivative,
$\nabla_x f$	x-gradient vector defined as $$(\nabla_x f)_i := \frac{\partial f}{\partial x_i},$$
ΔU	Laplace operator,
$\frac{\mathrm{D}f}{\mathrm{D}t}$	total (material) time derivative,
$f * g$	convolution of two functions,

$a \cdot b$	Euklidean scalar product of two vectors,
$\delta(x_1)$, δ^ε	Dirac's distribution in x_1-coordinate and its standard regularization,
$E_{\text{kin}}(f)$	total kinetic energy of the state f,
$E_{\text{pot}}(f)$	total potential energy of the state f,
$\|\| \cdot \|\|_{\text{pot}}$	norm derived from the potential energy defined as $$\|\|f\|\|_{\text{pot}} := \sqrt{-2E_{\text{pot}}(f)},$$
$\langle \cdot, \cdot \rangle_{\text{pot}}$	scalar product derived from the potential energy defined as $$\langle f, g \rangle := \iint \frac{f(x)g(y)}{\lvert x-y \rvert} \,\mathrm{d}x\,\mathrm{d}y,$$
$\mathcal{F}_M$	minimization class of function,
$\mathcal{H}$	total energy functional,
$\mathcal{C}$	Casimir functional,
$\mathcal{H}_\mathcal{C}$	energy-Casimir functional,
$\mathcal{H}_\mathcal{C}^{\mathrm{r}}$	reduced energy-Casimir functional,
$\mathcal{H}(f, \tilde{f})$	energy functional including dark matter distribution,
$\mathbf{M}, \mathbf{M}^{\mathrm{3D}}, \mathbf{M}^{\mathrm{FL}}$	combined constraint vector with ist flat and non-flat component,
h_M, h_M^{r}	infimum of the energy(-Casimir) and reduced energy-Casimir functional over appropriate set of functions

1 Introduction

One of the most classical problems in astrophysics is to describe an evolution of an ensemble of particles interacting among themselves through a force of some kind (gravity, magnetism, radiation, etc.). Particularly in galactic dynamics, where the number of particles can reach the order of 10^7–10^{12}, is important to choose a model which is a good approximation of the reality, which is mathematically well enough understood and of course it must be numerically computable in reasonable time. One of the most common non-relativistic setups uses non-radiating, electric neutral point masses. The equations of motion are then given by the Newton's Second law

$$m_j \ddot{\mathbf{q}}_j = G \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{m_j m_k (\mathbf{q}_j - \mathbf{q}_k)}{|\mathbf{q}_j - \mathbf{q}_k|^3}, \quad j = 1, \ldots, n,$$

where G denotes the universal gravitational constant and m_j, $\mathbf{q}_j$ mass and position vector of each individual particle. This second order system of ordinary differential equations together with proper initial conditions on positions $\mathbf{q}_j$ and velocities $\dot{\mathbf{q}}_j$ gives us the classical *N–body problem*. This model has, however, in the galactic scale several disadvantages. One of the biggest problems is the number of equations to be solved, which is (and remains in the nearest future) beyond the computational resources of contemporary computers. This problem is often overcome by reducing the number of equations using for example different methods of averaging.

The other option is to use statistical physics together with the theory of partial differential equations. Instead of describing a state of a system discretely for each individual particle (which is often undesirable, since the biggest interest lies on a global behavior of the system), we describe it globally as a density function f on a position-velocity phase-space. In the three-dimensional setting we have

$$f : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_0^+,$$

where

$$\iint_V f(t, x, v) \, \mathrm{d}x \, \mathrm{d}v$$

represents the mass contained in the phase-space volume V in the time t. The spatial density at position x is the sum over all velocities, i.e.

$$\rho_f(x) := \int f(t, x, v) \mathrm{d}v,$$

and the total mass of the system in the time t is given by

$$M := \iint f(t,x,v) \mathrm{d}x \, \mathrm{d}v.$$

When we suppose that there are no collisions among the particles, the density function f satisfies the so-called Liouville's theorem, which states that the distribution of particles in the phase-space is constant along the particle trajectories. That means, that the total derivative

$$\frac{\mathrm{D}f}{\mathrm{D}t} = 0.$$

When the particle trajectories $s \mapsto (X(s,t,x,v), V(s,t,x,v))$ obey the Newton's equations of motion

$$\begin{aligned} \dot{X}(s,t,x,v) &= V(s,t,x,v), && (1.1a) \\ \dot{V}(s,t,x,v) &= \mathbf{F}(s, X(s,t,x,v)), && (1.1b) \end{aligned}$$

the previous equation has in the Eulerian description the following form:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathbf{F} \cdot \nabla_v f = 0. \tag{1.2}$$

The equation (1.2) is called the *Vlasov equation* (or the collisionless Boltzmann equation). The vector $\mathbf{F}$ represents a force field, which drives the motion of the particles.

When we assume that the only force that acts on the system is gravitation created collectively by the particles, we can write

$$\mathbf{F} = -\nabla U,$$

where U is the gravitational potential. The Vlasov equation will now have the form

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0. \tag{1.3}$$

The gravitational potential U is in the non-relativistic case given as a solution of the Poisson equation

$$\Delta U = 4\pi \rho_f, \qquad \lim_{|x| \to \infty} U(x) = 0. \tag{1.4}$$

When we put the equations (1.3) and (1.4) together and supply suitable initial data f_0, we get the Vlasov-Poisson system:

$$\begin{aligned} \frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f &= 0, && (1.5a) \\ \Delta U &= 4\pi \rho_f, && (1.5b) \\ \lim_{|x| \to \infty} U(t,x) &= 0, && (1.5c) \\ f(0,x,v) &= f_0(x,v). && (1.5d) \end{aligned}$$

The Vlasov equation can be of course coupled with other types of field equations as well. If we want to investigate the evolution of particles in an electromagnetic field, we can use the Maxwell equations and we get the Vlasov-Maxwell system. The Vlasov-Einstein system results from a coupling with the Einstein gravitation equations and describes the particle evolution in the framework of General Relativity.

The existence theory differs strongly from one type of system to another and in case of the Vlasov-Poisson system, the global existence and uniqueness of a classical solution for initial data $f_0 \in C^1_c(\mathbb{R}^3)$ was proved in [20, 29]. The next sort of problems lies in the stability analysis of stationary solutions. This particular field is, especially in astrophysics, very important and receives a lot of attention. The results presented in this area originate primarily from the collaboration between Y. Guo and G. Rein.

The first necessary step in order to analyze stability of stationary solutions is to prove, that there are any stationary solutions. The strategy to construct them uses the conservation law of total mechanical energy. We define for a time-independent potential $U_0(x)$ the local (particle) energy E as

$$E(x, v) := \frac{1}{2}|v|^2 + U_0(x).$$

This energy is constant along the particle trajectories given by (1.1). Hence E and any function of E solves the Vlasov equation. It is therefore reasonable to search for stationary solutions f_0 in the form

$$f_0(x, v) = \phi(E(x, v)). \tag{1.6}$$

With this ansatz the Vlasov equation (1.5a) is satisfied and the spatial density ρ_{f_0} becomes a functional of the potential U_0. In order to obtain the self-consistent stationary solution of the Vlasov-Poisson system, we only need to solve (1.5b). If we find a solution to the semi-linear Poisson equation, then (1.6) defines a stationary solution of the Vlasov-Poisson system. It is natural, that only physically relevant solutions of this kind are allowed, for example the ones with finite total mass and with compact support.

We do not go into details concerning the existence of the general stationary solutions because the methods we use to prove stability of some of these solutions provide their existence automatically. To illustrate the variety of stationary solutions using different variations of (1.6) we give a few examples.

The ansatz

$$f_0(x, v) = (E_0 - E(x, v))^k_+, \quad E_0 < 0$$

leads for $-1 < k < 7/2$ to the so-called isotropic polytropes, spherically symmetric solutions with compact support and finite mass. Existence and stability of those solutions was proved in [15, 25, 26, 12]. The next class can be obtained if we allow dependence on

$$L(x, v) := |x \times v|^2,$$

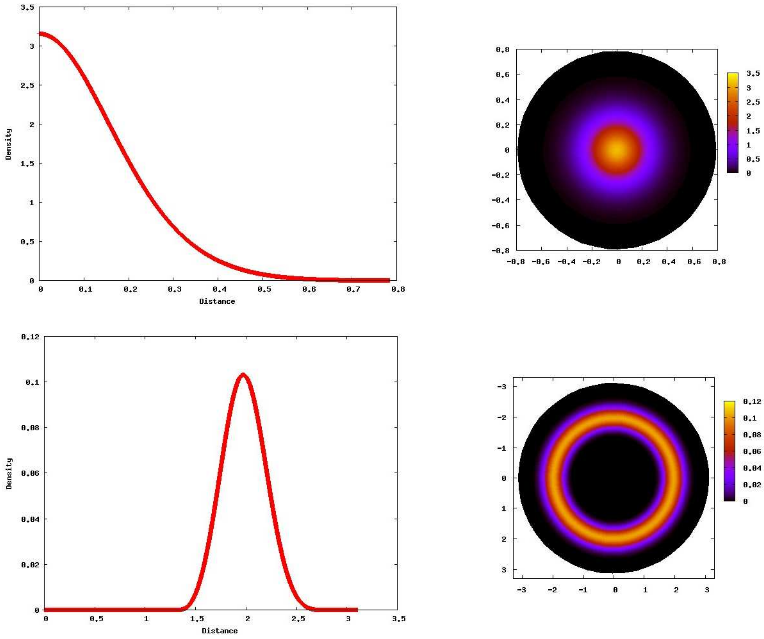

Figure 1.1: Example of the isotropic polytrope and stationary shell solution of the Vlasov-Poisson system (radial and planar density profiles).

which is the square of the modulus of angular momentum. If we are still in the spherically symmetric case, this quantity is also conserved along the particle trajectories (hence solves the Vlasov equation). The form of f_0 has in this case the form

$$f_0(x,v) = (E_0 - E(x,v))_+^k L(x,v)^l, \quad E_0 < 0,$$

and for $k > -1, l > -1, k + l + 1/2 > 0, k < 3l + 7/2$ we again get the stable stationary solutions of (1.5) (see [13]). The next (but definitely not last) type of stationary solutions uses the ansatz

$$f_0(x,v) = (E_0 - E(x,v))_+^k (L(x,v) - L_0)_+^l, \quad E_0 < 0, L_0 > 0.$$

This class of solutions (so-called stationary shells) was first introduced in [22] and the nonlinear stability was proved in [32].

All types of solutions mentioned above are spherically symmetric. Very little is known about the existence of steady-states with less symmetry. One example is discussed in [23], where axially symmetric solutions were obtained using the implicit function theorem as a perturbation to a spherically symmetric one.

It is an easy exercise to show that the total energy functional

$$\begin{aligned}\mathcal{H}(f) &:= E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f)\\ &= \frac{1}{2}\iint |v|^2 f(x,v)\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{2}\iint \frac{\rho_f(x)\rho_f(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{2}\iint |v|^2 f(x,v)\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{8\pi}||\nabla U_f||_2^2\end{aligned}$$

remains conserved as the solution evolves, which makes it a natural candidate for a Lyapunov function for the stability analysis. However, the Lyapunov approach works only when the stationary state is a critical point of the energy. But when we expand the functional $\mathcal{H}$ around some steady state f_0 (with potential U_0), we get

$$\mathcal{H}(f) - \mathcal{H}(f_0) = \iint \left(\frac{1}{2}|v|^2 + U_0(x)\right)(f - f_0)\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2\,\mathrm{d}x,$$

which has clearly non-vanishing linear part, hence f_0 cannot be a critical point of $\mathcal{H}$.

The flow $t \mapsto f(t)$ preserves not only the total energy, but the phase-space volume as well, hence all functionals in the form

$$\mathcal{C}(f) := \iint \Phi(f(x,v))\,\mathrm{d}x\,\mathrm{d}v \tag{1.7}$$

for $\Phi : \mathbb{R} \to \mathbb{R}$ reasonably smooth remain conserved along the flow. The functional (1.7) is called *Casimir functional* and it is an important tool in the stability analysis. When we now define the combined *energy-Casimir* functional as

$$\mathcal{H}_{\mathcal{C}}(f) := \mathcal{H}(f) + \mathcal{C}(f) \tag{1.8}$$

and expand it around some steady state

$$f_0 = \phi(E), \tag{1.9}$$

we obtain

$$\begin{aligned}\mathcal{H}_{\mathcal{C}}(f) &= \mathcal{H}_{\mathcal{C}}(f_0) + \iint \left(E + \Phi'(f_0)\right)(f - f_0)\,\mathrm{d}x\,\mathrm{d}v\\ &\quad -\frac{1}{8\pi}\int |\nabla U_f - \nabla U_{f_0}|^2\,\mathrm{d}x + \frac{1}{2}\iint \Phi''(f_0)(f-f_0)^2\,\mathrm{d}x\,\mathrm{d}v + \ldots.\end{aligned}$$

Now there is a chance, that if we (at least formally) put $\Phi' = -\phi^{-1}$, we can cancel the linear part out. That means that although f_0 is not a critical point of the energy functional $\mathcal{H}$, it is a critical point of the energy-Casimir functional $\mathcal{H}_{\mathcal{C}}$ for the properly defined Casimir functional. To obtain a stability result we expect the quadratic part in the expansion above to be definite. But for physically relevant steady states with finite mass and bounded support we must put ϕ in (1.9) strictly decreasing (ϕ^{-1} must exist) which implies $\Phi'' > 0$ and the definiteness of the quadratic part is lost.

To overcome this apparent failure of the energy-Casimir method, we reverse out approach. Instead of starting with an apriori given steady-state we start with an energy-Casimir functional and try to minimize it in some class of admissible functions. The minimizer, provided it exists, should be (as a critical point of the energy-Casimir functional) a steady-state of the Vlasov-Poisson system and its minimizing property hopefully gains some stability result.

2 The flat Vlasov–Poisson system

2.1 Disk-like galaxies

Most of the electromagnetic radiation emitted by a typical spiral galaxy comes from a thin disk. Therefore it is reasonable to ask, whether the Vlasov–Poisson theory can be modified, so that the dynamics of these very flattened objects can be investigated within its framework. This question, as it stands, is too complex to be answered in all its generality and a small portion of mathematical idealization is needed. We assume for simplicity, that the whole visible galactic matter is concentrated in an infinitesimally thin layer (in our case in the (x_1, x_2)–plane). To remain flat, we need the velocities to be concentrated on the (v_1, v_2)–plane as well. This means that the phase-space and the spatial densities have the form

$$\begin{aligned} f(t, \tilde{x}, x_3, \tilde{v}, v_3) &= \tilde{f}(t, \tilde{x}, \tilde{v})\delta(x_3)\delta(v_3), \\ \rho(t, \tilde{x}, x_3) &= \tilde{\rho}(t, \tilde{x})\delta(x_3), \end{aligned} \tag{2.1}$$

where δ denotes the classical Dirac distribution and $\tilde{x} := (x_1, x_2)$, $\tilde{v} := (v_1, v_2)$. This simplification is quite natural from a physical point of view. The objects in question are already flat enough (according to [11] the so-called "superthin" galaxies having an axial ratio over 8:1) and to make them totally flat does not make much of a difference. From the mathematical point of view, however, is the legitimacy of this approximation much more delicate and still, according to our knowledge, not fully understood (see more discussion on this topic in Chapter 6).

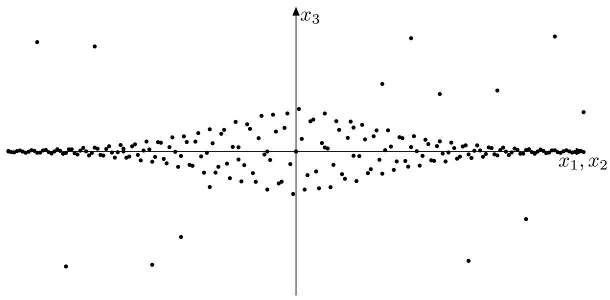

Figure 2.1: Typical side profile of a spiral galaxy

Now we have to construct the equations governing the evolution of such a flat ensemble of particles.

2.2 2D kinematics + 3D potential

The distribution function in the form (2.1) represents also a distributional solution of (1.2). We have for every test function $\varphi \in C_c^\infty(\mathbb{R}^6)$

$$\left\langle \frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathbf{F} \cdot \nabla_v f, \varphi \right\rangle_{\mathbb{R}^6} = 0.$$

If we now use the Green's theorem and expression (2.1) we get

$$\begin{aligned}
-\left\langle f, \frac{\partial \varphi}{\partial t} + v \cdot \nabla_x \varphi + \mathbf{F} \cdot \nabla_v \varphi \right\rangle_{\mathbb{R}^6} &= 0, \\
-\left\langle \tilde{f}\delta_{x_3}\delta_{v_3}, \frac{\partial \varphi}{\partial t} + v \cdot \nabla_x \varphi + \mathbf{F} \cdot \nabla_v \varphi \right\rangle_{\mathbb{R}^6} &= 0, \\
-\left\langle \tilde{f}, \left(\frac{\partial \varphi}{\partial t} + \tilde{v} \cdot \nabla_{\tilde{x}} \varphi + \tilde{\mathbf{F}} \cdot \nabla_{\tilde{v}} \varphi \right)\Big|_{(\tilde{x},0,\tilde{v},0)} \right\rangle_{\mathbb{R}^4} &= 0, \\
\left\langle \frac{\partial \tilde{f}}{\partial t} + \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{f} + \tilde{\mathbf{F}} \cdot \nabla_{\tilde{v}} \tilde{f}, \varphi(\tilde{x},0,\tilde{v},0) \right\rangle_{\mathbb{R}^4} &= 0.
\end{aligned}$$

We see here that the function f is a distributional solution of the three-dimensional Vlasov equation if and only if $\tilde{f}$ solves the two-dimensional Vlasov equation with the modified force term $\tilde{\mathbf{F}}$, which has in the gravitational case the form

$$\tilde{\mathbf{F}}(t,\tilde{x}) = -\int \frac{\tilde{x} - \tilde{y}}{|\tilde{x} - y|^3} \tilde{\rho}(\tilde{y}) \delta_{y_3} \, \mathrm{d}y = -\int \frac{\tilde{x} - \tilde{y}}{|\tilde{x} - \tilde{y}|^3} \tilde{\rho}(\tilde{y}) \, \mathrm{d}\tilde{y}.$$

When we simplify notations by dropping the tildes we obtain the following system of partial differential equations describing the evolution of such flat ensemble of particles:

$$\begin{aligned}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x U_f \cdot \nabla_v f &= 0, \qquad x, v \in \mathbb{R}^2, && (2.2a) \\
U(t,x) &= -\int \frac{\rho(y)}{|x-y|} \, \mathrm{d}y, && (2.2b) \\
\lim_{|x|\to\infty} U(t,x) &= 0, && (2.2c) \\
f(0,x,v) &= f_0(x,v). && (2.2d)
\end{aligned}$$

This system is called *flat Vlasov-Poisson system*. We can see that although particles occupy the two-dimensional domain, the gravitational interaction remains three-dimensional in nature with a typical $\frac{1}{r}$ singularity in the potential. This distinguishes (2.2) from the two-dimensional Vlasov-Poisson system, where the two-dimensional logarithmic gravitational potential appears. The latter system can be used to model hypothetical infinitely long cylindrical objects (see [6]).

The main difficulty in the analysis of (2.2) is that the potential is "more singular" in dimension two than in regular three-dimensional case. For example the force kernel $\frac{x}{|x|^3}$ is not even local integrable in $\mathbb{R}^2$. As regards the existence theory we have "only" existence (without uniqueness) of local classical solution and global weak solution (both proved in [5]), compared to the global unique classical solution for (1.5). The general existence and uniqueness theory for singular solutions of (1.5) was established using the Coulomb algebras was established in [14], but this concept seems to be too abstract to be used in our stability analysis.

To analyze stability of stationary solutions of the flat Vlasov-Poisson system we modify the variational energy-Casimir method described in Chapter 1.

3 Stability via reduction

The aim of this chapter is to prove the existence of a large class of non-linearly stable steady states of the flat Vlasov-Poisson system. To do so we follow the approach developed by Guo and Rein [15, 16, 17] in the regular, three dimensional situation. We prove that under suitable assumptions on a prescribed function $\Phi : [0,\infty[\to [0,\infty[$ the energy-Casimir functional

$$\begin{aligned}\mathcal{H}_{\mathcal{C}}(f) &= \frac{1}{2}\iint |v|^2 f(x,v)\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2}\iint\iint \frac{f(x,v)\,f(y,w)}{|x-y|}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}y\\ &\quad + \iint \Phi(f(x,v))\,\mathrm{d}v\,\mathrm{d}x\end{aligned}$$

has a minimizer f_0 subject to the constraint

$$\iint f(x,v)\,\mathrm{d}v\,\mathrm{d}x = M,$$

where $M > 0$, the total mass of the resulting steady state, is prescribed.

In [21] this approach has already been used to construct stable steady states of the flat Vlasov-Poisson system. Here we obtain a number of improvements and extensions of this earlier result. Firstly, we use a reduction procedure for proving the existence of a minimizer of $\mathcal{H}_{\mathcal{C}}$. This approach is mathematically more elegant and adequate, since the reduced functional lives on the set of spatial densities ρ, and the main difficulty in the variational problem lies in the potential energy part which does not really depend on f but only on the spatial density induced by f. More importantly, the reduced variational problem is of interest in its own right since it provides a stability result for the flat Euler-Poisson system which is the fluid dynamical analogue of the kinetic Vlasov-Poisson system. For the reduction procedure to work the function Φ has to satisfy certain growth conditions. An example of a steady state which violates this growth condition is the so-called Kuzmin disk which is known in the astrophysics literature and was not covered by previous results. The Kuzmin disk is closely investigated in Chapter 4. Secondly, in [21] the perturbations admissible in the stability result had to be supported on the plane and in addition had to be spherically symmetric. In this thesis we remove the latter, unphysical restriction. It is desirable to remove also the restriction that the perturbations have to live in the plane, but that is much harder and is still under investigation (see the discussion on that topic in Chapter 6). Lastly, we relax the assumptions on Φ the main one being that Φ be strictly convex so that we cover a larger class of steady states, and we obtain stability estimates in stronger norms than were obtained previously.

This chapter proceeds as follows. In the next section we introduce various functionals and the reduced version of the variational problem, and we establish the connection between the original and the reduced variational problem. In Section 3.2 we establish the existence of a minimizer to the reduced problem using a concentration-compactness argument; notice that the variational problem—both reduced and original—is non-trivial since the energy-Casimir functional is not convex and is defined on functions supported on $\mathbb{R}^2$ or $\mathbb{R}^4$ respectively. In Section 3.3 we derive our stability result, and in the Section 3.4 we consider the stability result for the Euler-Poisson system which arises from the reduced functional.

3.1 Energy–Casimir functionals and reduction

For $\rho = \rho(x)$ measurable we define the induced gravitational potential and potential energy as

$$U_\rho(x) := -\int \frac{\rho(y)}{|x-y|}\,\mathrm{d}y,$$

$$E_{\mathrm{pot}}(\rho) := \frac{1}{2}\int U_\rho(x)\rho(x)\,\mathrm{d}x = -\frac{1}{2}\iint \frac{\rho(x)\rho(y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x;$$

the integrals $\int$ without a subscript always (except Chapter 5) extend over $\mathbb{R}^2$. It will also be useful to introduce the bilinear form which corresponds to the potential energy, i.e., for $\rho, \sigma : \mathbb{R}^2 \to \mathbb{R}$ measurable,

$$\langle \rho, \sigma \rangle_{\mathrm{pot}} := \frac{1}{2}\iint \frac{\rho(x)\sigma(y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x,$$

so that in particular $E_{\mathrm{pot}}(\rho) = -\langle \rho, \rho \rangle_{\mathrm{pot}}$. For the convenience of the reader we collect the main estimates for potentials, potential energies, and the above bilinear form, which we will need.

Lemma 3.1. *If $\rho \in L^{4/3}(\mathbb{R}^2)$, then $U_\rho \in L^4(\mathbb{R}^2)$, and there exists a constant $C > 0$ such that for all $\rho \in L^{4/3}(\mathbb{R}^2)$ the estimates*

$$||U_\rho||_4 \le C||\rho||_{4/3}, \quad -E_{\mathrm{pot}}(\rho) \le C||\rho||_{4/3}^2$$

hold. The bilinear form $\langle \cdot, \cdot \rangle_{\mathrm{pot}}$ defines a scalar product on $L^{4/3}(\mathbb{R}^2)$ with induced norm

$$||\rho||_{\mathrm{pot}} := \langle \rho, \rho \rangle_{\mathrm{pot}}^{1/2} = (-E_{\mathrm{pot}}(\rho))^{1/2},$$

in particular,

$$\langle \rho, \sigma \rangle_{\mathrm{pot}} \le (E_{\mathrm{pot}}(\rho)\,E_{\mathrm{pot}}(\sigma))^{1/2} = ||\rho||_{\mathrm{pot}}||\sigma||_{\mathrm{pot}}.$$

Proof. Since $1/|\cdot| \in L^2_w(\mathbb{R}^2)$, the weak L^2 space, the assertions on U_ρ follow by the generalized Young's inequality [19, 4.3]. The estimate for the potential energy is nothing but the Hardy-Littlewood-Sobolev inequality [19, 4.3] and follows by Hölder's inequality, and so does the fact that $\langle \cdot, \cdot \rangle_{\mathrm{pot}}$ is defined on $L^{4/3}(\mathbb{R}^2)$. The positive definiteness of $\langle \cdot, \cdot \rangle_{\mathrm{pot}}$ can be shown exactly like the positivity of the Coulomb energy in the three dimensional case, cf. [19, 9.8]. □

Let $f = f(x, v)$ be a measurable function on phase space. We define the induced spatial density, gravitational potential, and potential energy as

$$\rho_f(x) := \int f(x, v)\, \mathrm{d}v, \quad U_f := U_{\rho_f}, \quad E_{\mathrm{pot}}(f) := E_{\mathrm{pot}}(\rho_f).$$

In addition, we define the kinetic energy

$$E_{\mathrm{kin}}(f) := \frac{1}{2} \iint |v|^2 f(x, v)\, \mathrm{d}v\, \mathrm{d}x,$$

the so-called Casimir functional

$$\mathcal{C}(f) := \iint \Phi(f(x, v))\, \mathrm{d}v\, \mathrm{d}x$$

with $\Phi : [0, \infty[\to [0, \infty[$ prescribed, and the energy-Casimir functional

$$\mathcal{H}_{\mathcal{C}}(f) := \mathcal{C}(f) + E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f).$$

The total energy $E_{\mathrm{kin}} + E_{\mathrm{pot}}$ as well as the Casimir functional $\mathcal{C}$ and hence also their sum $\mathcal{H}_{\mathcal{C}}$ are conserved along sufficiently regular solutions of the flat Vlasov-Poisson system. As regards Φ, we assume for the moment that

$$\Phi \in C^1([0, \infty[) \text{ is strictly convex, } \Phi(0) = \Phi'(0) = 0, \ \lim_{\eta \to \infty} \Phi(\eta)/\eta = \infty.$$

These assumptions make Φ non-negative and Φ' a bijection on $[0, \infty[$.

Our aim is to show that the energy-Casimir functional $\mathcal{H}_{\mathcal{C}}$ has a minimizer in the constraint set

$$\mathcal{F}_M := \left\{ f \in L^1_+(\mathbb{R}^4) \mid E_{\mathrm{kin}}(f) + \mathcal{C}(f) < \infty, \ \rho_f \in L^{4/3}(\mathbb{R}^2), \iint f = M \right\},$$

where $M > 0$ is prescribed, and the subscript $+$ indicates that only non-negative functions are considered. Since the troublesome term in the functional is the potential energy which actually depends only on the spatial density induced by f we introduce a reduced variational problem for a functional which is defined in terms of spatial densities ρ. For $r \geq 0$ we define

$$\mathcal{G}_r := \left\{ g \in L^1_+(\mathbb{R}^2) \mid \int \left(\frac{1}{2}|v|^2 g(v) + \Phi(g(v)) \right) \mathrm{d}v < \infty, \int g(v)\, \mathrm{d}v = r \right\}$$

and

$$\Psi(r) := \inf_{g \in \mathcal{G}_r} \int \left(\frac{1}{2}|v|^2 g(v) + \Phi(g(v))\right) \mathrm{d}v.$$

The idea behind this construction is to first minimize the energy-Casimir functional over all functions $f(x,v)$ which upon integration in v give the same spatial density ρ, and then minimize with respect to the latter in a second (and main) step. This approach was introduced in [24, 33].

The reduced variational problem is to minimize the reduced functional

$$\mathcal{H}_{\mathcal{C}}^{\mathrm{r}}(\rho) := \int \Psi(\rho(x))\,\mathrm{d}x + E_{\mathrm{pot}}(\rho)$$

over the set

$$\mathcal{F}_M^{\mathrm{r}} := \left\{\rho \in L^{4/3} \cap L^1_+(\mathbb{R}^2) \mid \int \Psi(\rho(x))\,\mathrm{d}x < \infty, \int \rho(x)\,\mathrm{d}x = M\right\}.$$

We need to establish a relation between minimizers of the original functional and minimizers of the reduced one. Here we can essentially follow the corresponding results proved for the three dimensional case in [24]. First of all we explore the relation between Φ and Ψ. For a function $h : \mathbb{R} \to]-\infty, \infty]$ we denote by

$$h^*(\lambda) := \sup_{r \in \mathbb{R}} (\lambda r - h(r))$$

its Legendre transform. In what follows constants denoted by C are always positive, may depend on Φ and M, and may change their value from line to line.

Lemma 3.2. *Let Φ and Ψ be as specified respectively defined above, and extend both functions by $+\infty$ to the interval $]-\infty, 0]$. Then the following holds:*

(a) *For $\lambda \in \mathbb{R}$,*

$$\Psi^*(\lambda) = \int \Phi^*\left(\lambda - \frac{1}{2}|v|^2\right) \mathrm{d}v,$$

and in particular $\Phi^*(\lambda) = 0 = \Psi^*(\lambda)$ for all $\lambda < 0$.

(b) *$\Psi \in C^1([0,\infty[)$ is strictly convex, and $\Psi(0) = \Psi'(0) = 0$.*

(c) *Let $k > 0$ and $n = k + 1$.*

  (i) *If $\Phi(f) = C f^{1+1/k}$ for $f \geq 0$, then $\Psi(\rho) = C\rho^{1+1/n}$ for $\rho \geq 0$.*
  (ii) *If $\Phi(f) \geq C f^{1+1/k}$ for $f \geq 0$ large, then $\Psi(\rho) \geq C\rho^{1+1/n}$ for $\rho \geq 0$ large.*
  (iii) *If $\Phi(f) \leq C f^{1+1/k}$ for $f \geq 0$ small, then $\Psi(\rho) \leq C\rho^{1+1/n}$ for $\rho \geq 0$ small.*

If the restriction to large or small values of f can be dropped then the corresponding restriction for ρ can be dropped as well.

Proof. By definition

$$\begin{aligned}
\Psi^*(\lambda) &= \sup_{r\geq 0}\left[\lambda r - \inf_{r\in\mathcal{G}_r}\int\left(\frac{1}{2}|v|^2 g(v) + \Phi(g(v))\right)\,\mathrm{d}v\right]\\
&= \sup_{r\geq 0}\sup_{g\in\mathcal{G}_r}\int\left[\left(\lambda-\frac{1}{2}|v|^2\right)g(v) - \Phi(g(v))\right]\,\mathrm{d}v\\
&= \sup_{g\in L^1_+(\mathbb{R}^2)}\int\left[\left(\lambda-\frac{1}{2}|v|^2\right)g(v) - \Phi(g(v))\right]\,\mathrm{d}v\\
&\leq \int\sup_{y\geq 0}\left[\left(\lambda-\frac{1}{2}|v|^2\right)y - \Phi(y)\right]\,\mathrm{d}v = \int\Phi^*\left(\lambda-\frac{1}{2}|v|^2\right)\,\mathrm{d}v.
\end{aligned}$$

For $\lambda \leq 0$ both sides of this estimate are zero, so consider $\lambda > 0$. If $|v| \geq \sqrt{2\lambda}$ then $\sup_{y\geq 0}[\cdots] = 0$ and for $|v| < \sqrt{2\lambda}$ the supremum is attained at $y = y_v := (\Phi')^{-1}(\lambda - \frac{1}{2}|v|^2)$. Hence with the definition

$$g_0(v) := \begin{cases} y_v & \text{for} \quad |v| < \sqrt{2\lambda} \\ 0 & \text{for} \quad |v| \geq \sqrt{2\lambda} \end{cases},$$

we obtain the reversed estimate, and part (a) is established. Part (b) is standard for Legendre transforms, and we refer to [24, Lemma 2.2] for the details. As to (c), if we assume that $\Phi(f) \geq Cf^{1+1/k}$ for $f \geq 0$ large, we find that $\Phi(f) \geq Cf^{1+1/k} - C'$ for $f \geq 0$. Hence for $\lambda \geq 0$,

$$\Phi^*(\lambda) = \sup_{f\geq 0}(\lambda f - \Phi(f)) \leq C' + \sup_{f\geq 0}\left(\lambda f - Cf^{1+\frac{1}{k}}\right) \leq C + C\,\lambda^{k+1},$$

and

$$\begin{aligned}
\Psi^*(\lambda) = \int_{|v|\leq\sqrt{2\lambda}}\Phi^*\left(\lambda-\frac{1}{2}|v|^2\right)\,\mathrm{d}v &\leq C\int_{|v|\leq\sqrt{2\lambda}}\left[1+\left(\lambda-\frac{1}{2}|v|^2\right)^{k+1}\right]\,\mathrm{d}v\\
\leq C\lambda + C\int_{|v|\leq\sqrt{2\lambda}}\left(\lambda-\frac{1}{2}|v|^2\right)^{k+1}\,\mathrm{d}v &\leq C + C\lambda^{k+2} = C + C\lambda^{1+n}.
\end{aligned}$$

Using the fact that $\Psi^{**} = \Psi$ we obtain the estimate

$$\Psi(\rho) = \sup_{\lambda\geq 0}(\rho\lambda - \Psi^*(\lambda)) \geq -C' + \sup_{\lambda\geq 0}(\rho\lambda - C\lambda^{1+n}) = C\rho^{1+1/n} - C',$$

which proves (c) (ii). The remaining estimates are shown in a similar way. □

The relation between the minimizers of $\mathcal{H}_\mathcal{C}$ and $\mathcal{H}^{\mathrm{r}}_\mathcal{C}$ is as follows.

Theorem 3.3. (a) *For every function $f \in \mathcal{F}_M$,*

$$\mathcal{H}_{\mathcal{C}}(f) \geq \mathcal{H}_{\mathcal{C}}^{\mathrm{r}}(\rho_f),$$

with equality if f is a minimizer of $\mathcal{H}_{\mathcal{C}}$ over $\mathcal{F}_M$.

(b) *Let $\rho_0 \in \mathcal{F}_M^{\mathrm{r}}$ be a minimizer of $\mathcal{H}_{\mathcal{C}}^{\mathrm{r}}$ with induced potential U_0. Then there exists a Lagrange multiplier $E_0 \in \mathbb{R}$ such that the identity*

$$\rho_0 = \begin{cases} (\Psi')^{-1}(E_0 - U_0) & , \quad U_0 < E_0 \\ 0 & , \quad U_0 \geq E_0 \end{cases}$$

holds almost everywhere. The function

$$f_0 := \begin{cases} (\Phi')^{-1}(E_0 - E) & , \quad E < E_0 \\ 0 & , \quad E \geq E_0 \end{cases} \quad \text{with } E = E(x,v) := \frac{1}{2}|v|^2 + U_0(x)$$

is a minimizer of $\mathcal{H}_{\mathcal{C}}$ in $\mathcal{F}_M$.

Proof. Since the proof follows the same lines as [24, Thm 2.1] we only indicate the main arguments. The estimate in part (a) follows directly from the definitions. Next one can show that if $f \in \mathcal{F}_M$ is such that up to sets of measure zero,

$$\Phi'(f) = E_0 - E > 0 \text{ where } f > 0, \text{ and } E_0 - E \leq 0 \text{ where } f = 0. \tag{3.1}$$

with E defined as in (b) but with U_f instead of U_0 and E_0 a constant, then equality holds in part (a). If f is a minimizer of $\mathcal{H}_{\mathcal{C}}$, then the Euler-Lagrange equation implies that f is of the above form for some Lagrange multiplier E_0, and equality holds in (a). The relation of ρ_0 and U_0 in part (b) is nothing but the Euler-Lagrange equation for the reduced variational problem. If f_0 is defined as in (b) then $\rho_0 = \rho_{f_0}$, in particular, $f_0 \in \mathcal{F}_M$, and (3.1) holds by definition of f_0. Hence equality holds in (a) for f_0 so that by part (a) for any other $f \in \mathcal{F}_M$,

$$\mathcal{H}_{\mathcal{C}}(f) \geq \mathcal{H}_{\mathcal{C}}^{\mathrm{r}}(\rho_f) \geq \mathcal{H}_{\mathcal{C}}^{\mathrm{r}}(\rho_0) = \mathcal{H}_{\mathcal{C}}(f_0),$$

which means that f_0 minimizes $\mathcal{H}_{\mathcal{C}}$. □

Remark. (a) In the next section we show that under suitable assumptions on Ψ which can be translated into corresponding assumptions on Φ the reduced variational problem has a solution ρ_0. The minimizer f_0 obtained by the lifting procedure in part (b) of the theorem depends only on the particle energy E. The latter is for the time-independent potential U_0 constant along characteristics of the Vlasov equation, and hence f_0 is a steady state of the flat Vlasov-Poisson system.

(b) If $\mathcal{H}_{\mathcal{C}}^{\mathrm{r}}$ has at least one minimizer in $\mathcal{F}_M^{\mathrm{r}}$ and if $f_0 \in \mathcal{F}_M$ is a minimizer of $\mathcal{H}_{\mathcal{C}}$, then one can show that $\rho_0 := \rho_{f_0} \in \mathcal{F}_M^{\mathrm{r}}$ is a minimizer of $\mathcal{H}_{\mathcal{C}}^{\mathrm{r}}$. This map is one-to-one between the sets of minimizers of $\mathcal{H}_{\mathcal{C}}$ in $\mathcal{F}_M$ and $\mathcal{H}_{\mathcal{C}}^{\mathrm{r}}$ in $\mathcal{F}_M^{\mathrm{r}}$ and is inverse to the mapping $\rho_0 \mapsto f_0$ described in part (b) of the theorem.

3.2 Existence of a solution to the reduced variational problem

In the present section we prove that the reduced energy-Casimir functional $\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}$ has a minimizer in the constraint set

$$\mathcal{F}^{\mathrm{r}}_M := \left\{ \rho \in L^1_+(\mathbb{R}^2) \mid \int \Psi(\rho(x))\,\mathrm{d}x < \infty, \int \rho(x)\,\mathrm{d}x = M \right\},$$

where $M > 0$ is prescribed and Ψ satisfies the assumptions $\Psi \in C^1([0,\infty[)$, $\Psi(0) = \Psi'(0) = 0$ and

(Ψ1) Ψ is strictly convex,

(Ψ2) $\Psi(\rho) \geq C\rho^{1+1/n}$ for $\rho \geq 0$ large,

(Ψ3) $\Psi(\rho) \leq C\rho^{1+1/n'}$ for $\rho \geq 0$ small,

with growth rates $n, n' \in]0,2[$. The core of the proof is a concentration-compactness argument to show that along a minimizing sequence the matter cannot spread out but has to remain concentrated in a finite region of space. First however we show that the energy-Casimir functional is bounded from below in such a way that minimizing sequences are bounded in a suitable L^p space.

Lemma 3.4. *Under the above assumptions on Ψ and for $\rho \in \mathcal{F}^{\mathrm{r}}_M$,*

$$\int \rho^{1+1/n}\,\mathrm{d}x \leq C + C\int \Psi(\rho)\,\mathrm{d}x,$$

and

$$\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\rho) \geq \int \Psi(\rho)\,\mathrm{d}x - C - C\left(\int \Psi(\rho)\,\mathrm{d}x\right)^{n/2}.$$

In particular,

$$h^{\mathrm{r}}_M := \inf_{\mathcal{F}^{\mathrm{r}}_M} \mathcal{H}^{\mathrm{r}}_{\mathcal{C}} > -\infty.$$

Proof. The first estimate follows by assumption (Ψ2) and the fact that $\int \rho = M$. By Lemma 3.1 and interpolation,

$$-E_{\mathrm{pot}}(\rho) \leq C||\rho||^2_{4/3} \leq C||\rho||_1^{(3-n)/2}||\rho||_{1+1/n}^{(n+1)/2} \leq C + C\left(\int \Psi(\rho)\,\mathrm{d}x\right)^{n/2},$$

and since $0 < n < 2$ the proof is complete. □

We note an immediate corollary.

Corollary 3.5. *Any minimizing sequence of $\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}$ in $\mathcal{F}^{\mathrm{r}}_M$ is bounded in $L^{1+1/n}(\mathbb{R}^2)$ and therefore contains a subsequence which converges weakly in that space.*

The concentration-compactness argument mentioned above relies on the behavior of $\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\rho)$ if ρ is scaled or split into several parts. We start with the latter; in the sequel B_R denotes the open ball of radius $R > 0$ about the origin.

Lemma 3.6. *Let $\rho \in \mathcal{F}^{\mathrm{r}}_M$. Then for $R > 1$,*

$$\sup_{a\in\mathbb{R}^2} \int_{a+B_R} \rho(x)\,\mathrm{d}x \geq \frac{1}{RM}\left(-2E_{\mathrm{pot}}(\rho) - M^2R^{-1} - C||\rho||^2_{1+1/n}R^{-(3-n)/(n+1)}\right).$$

Proof. We split the potential energy as follows:

$$\begin{aligned} -2E_{\mathrm{pot}} &= \iint_{|x-y|\leq 1/R} \frac{\rho(x)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y + \iint_{1/R<|x-y|<R} \cdots + \iint_{|x-y|\geq R} \cdots \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

By Hölder's and Young's inequalities we obtain estimates

$$\begin{aligned} I_1 &\leq ||\rho||_{1+1/n}||\rho * (\mathbf{1}_{B_{1/R}}1/|\cdot|)||_{n+1} \leq ||\rho||^2_{1+1/n}||\mathbf{1}_{B_{1/R}}1/|\cdot|||_{(n+1)/2} \\ &\leq C||\rho||^2_{1+1/n}R^{-(3-n)/(n+1)}, \\ I_2 &\leq R\iint_{|x-y|\leq R} \rho(x)\rho(y)\,\mathrm{d}x\,\mathrm{d}y = MR\sup_{a\in\mathbb{R}^2}\int_{a+B_R}\rho(x)\,\mathrm{d}x, \\ I_3 &\leq M^2R^{-1}. \end{aligned}$$

We insert these estimates into the formula for $-2E_{\mathrm{pot}}$ and rearrange terms to obtain the assertion. □

Next we investigate the behavior of the reduced functional under scalings.

Lemma 3.7. (a) *For every $M > 0$, $h^{\mathrm{r}}_M < 0$.*

(b) *For every $0 < \overline{M} \leq M$ the estimate $h^{\mathrm{r}}_{\overline{M}} \geq (\overline{M}/M)^{3/2}h^{\mathrm{r}}_M$ holds.*

Proof. For $\rho \in \mathcal{F}^{\mathrm{r}}_M$ and $a, b > 0$ we define $\bar{\rho}(x) := a\rho(bx)$. Then

$$\begin{aligned} \int \bar{\rho}\,\mathrm{d}x &= ab^{-2}\int\rho\,\mathrm{d}x, \\ E_{\mathrm{pot}}(\bar{\rho}) &= a^2b^{-3}E_{\mathrm{pot}}(\rho), \\ \int\Psi(\bar{\rho})\,\mathrm{d}x &= b^{-2}\int\Psi(a\rho)\,\mathrm{d}x. \end{aligned}$$

To prove part (a) we fix a bounded and compactly supported function $\rho \in \mathcal{F}^{\mathrm{r}}_M$ and choose $a = b^2$ so that $\bar{\rho} \in \mathcal{F}^{\mathrm{r}}_M$ as well. By $(\Psi 3)$ and since $2/n' > 1$,

$$\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\bar{\rho}) = b^{-2}\int\Psi(b^2\rho)\,\mathrm{d}x + bE_{\mathrm{pot}}(\rho) \leq Cb^{2/n'} + bE_{\mathrm{pot}}(\rho) < 0$$

for b sufficiently small, and part (a) is established. As to part (b), we take $a = 1$ and $b = (M/\overline{M})^{1/2} \geq 1$. For this choice of parameters the mapping $\mathcal{F}^{\mathrm{r}}_M \ni \rho \mapsto \bar\rho \in \mathcal{F}^{\mathrm{r}}_{\overline{M}}$ is one-to-one and onto, and the estimate

$$\begin{aligned}\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\bar\rho) &= b^{-2}\int \Psi(\rho)\,\mathrm{d}x + b^{-3}E_{\mathrm{pot}}(\rho)\\ &\geq b^{-3}\left(\int \Psi(\rho)\,\mathrm{d}x + E_{\mathrm{pot}}(\rho)\right) = \left(\frac{\overline{M}}{M}\right)^{3/2}\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\rho)\end{aligned}$$

proves the assertion of part (b). □

Corollary 3.8. *Let $(\rho_i) \subset \mathcal{F}^{\mathrm{r}}_M$ be a minimizing sequence of $\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}$. Then there exist $\delta_0 > 0$, $R_0 > 0$, and a sequence of shift vectors $(a_i) \subset \mathbb{R}^2$ such that for i sufficiently large,*

$$\int_{a_i + B_{R_0}} \rho_i(x)\,\mathrm{d}x \geq \delta_0.$$

Proof. By Corollary 3.5, $(||\rho_i||_{1+1/n})$ is bounded. By Lemma 3.7 (a),

$$E_{\mathrm{pot}}(\rho_i) \leq \mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\rho_i) \leq \frac{1}{2}h^{\mathrm{r}}_M < 0$$

for i sufficiently large, and the assertion follows by Lemma 3.6. □

This corollary only shows that along a minimizing sequence not all matter can spread uniformly. In the proof of the existence theorem below we shall actually see that the matter remains within a ball of finite radius up to spatial shifts and an arbitrarily small remainder. In such a situation we have the following compactness result:

Lemma 3.9. *Let $(\rho_i) \subset \mathcal{F}^{\mathrm{r}}_M$ be such that*

$$\rho_i \rightharpoonup \rho_0 \text{ weakly in } L^{1+1/n}(\mathbb{R}^2)$$

and such that the following concentration property holds:

$$\forall \epsilon > 0\ \exists R > 0:\ \limsup_{i\to\infty}\int_{|x|>R} \rho_i(x)\,\mathrm{d}x < \epsilon.$$

Then

$$E_{\mathrm{pot}}(\rho_i - \rho_0) \to 0 \text{ and } E_{\mathrm{pot}}(\rho_i) \to E_{\mathrm{pot}}(\rho_0),\ i \to \infty.$$

Proof. By weak convergence $\rho_0 \geq 0$ a. e., and $\int \rho_0 \leq M$. We define $\sigma_i := \rho_i - \rho_0$ so that $\sigma_i \rightharpoonup 0$ weakly in $L^{1+1/n}(\mathbb{R}^2)$, the concentration property holds for $|\sigma_i|$ as well, and $\int |\sigma_i| \leq 2M$. We need to prove that

$$I_i := \iint \frac{\sigma_i(x)\sigma_i(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \to 0,$$

which is the first assertion. Since

$$E_{\text{pot}}(\rho_i) - E_{\text{pot}}(\rho_0) = E_{\text{pot}}(\rho_i - \rho_0) + \int U_{\rho_0}(\rho_i - \rho_0),$$

the fact that $U_{\rho_0} \in L^4(\mathbb{R}^2)$ together with the weak convergence of ρ_i implies the second assertion. For $\delta > 0$ and $R > 0$ we split the domain of integration into three subsets defined by

$$\begin{aligned}
&|x-y| < \delta, \\
&|x-y| \geq \delta \wedge (|x| \geq R \vee |y| \geq R), \\
&|x-y| \geq \delta \wedge |x| < R \wedge |y| < R,
\end{aligned}$$

and we denote the corresponding contributions to I_i by $I_{i,1}$, $I_{i,2}$, $I_{i,3}$. Young's inequality implies that

$$|I_{i,1}| \leq C||\sigma_i||^2_{1+1/n}||\mathbf{1}_{B_\delta}|\cdot|^{-1}||_{(n+1)/2} \leq C\left(\int_0^\delta r^{(1-n)/2}\,\mathrm{d}r\right)^{2/(n+1)}$$

which can be made as small as we wish, uniformly in i and $R > 0$, by making $\delta > 0$ small. For $\delta > 0$ now fixed,

$$|I_{i,2}| \leq \frac{4M}{\delta}\int_{|x|>R}|\sigma_i(x)|\,\mathrm{d}x$$

which becomes small for $i \to \infty$ by the concentration assumption, if we choose $R > 0$ accordingly. Finally by Hölder's inequality,

$$|I_{i,3}| = \left|\int \sigma_i(x)h_i(x)\,\mathrm{d}x\right| \leq ||\sigma_i||_{1+1/n}||h_i||_{1+n} \leq C||h_i||_{1+n},$$

where in a pointwise sense,

$$h_i(x) := \mathbf{1}_{B_R}(x)\int_{|x-y|\geq\delta}\mathbf{1}_{B_R}(y)\frac{1}{|x-y|}\sigma_i(y)\,\mathrm{d}y \to 0$$

due to the weak convergence of σ_i and the fact that the test function against which σ_i is integrated here is in L^{1+n}. Since $|h_i| \leq \frac{2M}{\delta}\mathbf{1}_{B_R}$ uniformly in i, Lebesgue's dominated convergence theorem implies that $h_i \to 0$ in L^{1+n}, and the proof is complete. □

We have now assembled all the tools we need to prove the existence of a minimizer of the reduced functional.

Theorem 3.10. *Let $(\rho_i) \subset \mathcal{F}^{\mathrm{r}}_M$ be a minimizing sequence of $\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}$. Then there exists a sequence of shift vectors $(a_i) \subset \mathbb{R}^2$ and a subsequence, again denoted by (ρ_i), such that for every $\varepsilon > 0$ there exist $R > 0$ with*

$$\int_{a_i+B_R}\rho_i(x)\,\mathrm{d}x \geq M - \varepsilon,\ \ i \in \mathbb{N},$$

$$T_{a_i}\rho_i := \rho_i(\cdot + a_i) \rightharpoonup \rho_0 \text{ weakly in } L^{1+1/n}(\mathbb{R}^2),\ i \to \infty,$$

$$\int_{B_R} \rho_0(x)\,\mathrm{d}x \geq M - \varepsilon.$$

Finally,

$$E_{\mathrm{pot}}(T_{a_i}\rho_i - \rho_0) \to 0,$$

and $\rho_0 \in \mathcal{F}^{\mathrm{r}}_M$ *is a minimizer of* $\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}$*.*

Proof. We split $\rho \in \mathcal{F}^{\mathrm{r}}_M$ as follows:

$$\rho = \mathbf{1}_{B_{R_1}}\rho + \mathbf{1}_{B_{R_2}\setminus B_{R_1}}\rho + \mathbf{1}_{\mathbb{R}^2\setminus B_{R_2}}\rho =: \rho_1 + \rho_2 + \rho_3.$$

The parameters $R_1 < R_2$ of the split are yet to be determined. Recalling the definition of the bilinear form $\langle\cdot,\cdot\rangle_{\mathrm{pot}}$,

$$\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\rho) = \mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\rho_1) + \mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\rho_2) + \mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\rho_3) - 2\langle\rho_1+\rho_3,\rho_2\rangle_{\mathrm{pot}} - 2\langle\rho_1,\rho_3\rangle_{\mathrm{pot}}.$$

If we choose $R_2 > 2R_1$, then

$$\langle\rho_1,\rho_3\rangle_{\mathrm{pot}} \leq \frac{C}{R_2}.$$

By Lemma 3.1 and interpolation,

$$\begin{aligned}\langle\rho_1+\rho_3,\rho_2\rangle_{\mathrm{pot}} &\leq ||\rho_1+\rho_3||_{\mathrm{pot}}||\rho_2||_{\mathrm{pot}}\\ &\leq C\,||\rho_1+\rho_3||_{4/3}||\rho_2||_{\mathrm{pot}} \leq C\,||\rho||_{1+1/n}^{(n+1)/4}||\rho_2||_{\mathrm{pot}}.\end{aligned}$$

If we define

$$M_l := \int \rho_l(x)\,\mathrm{d}x,\ l = 1,2,3,$$

then Lemma 3.7 (b) and the estimates above imply that

$$\begin{aligned}h^{\mathrm{r}}_M - \mathcal{H}^{\mathrm{r}}_{\mathcal{C}}(\rho) &\leq \left(1 - \left(\frac{M_1}{M}\right)^{3/2} - \left(\frac{M_2}{M}\right)^{3/2} - \left(\frac{M_3}{M}\right)^{3/2}\right)h^{\mathrm{r}}_M\\ &\quad + C\left(R_2^{-1} + ||\rho||_{1+1/n}^{(n+1)/4}||\rho_2||_{\mathrm{pot}}\right)\\ &\leq \frac{C}{M^2}(M_1M_2 + M_2M_3 + M_1M_3)h^{\mathrm{r}}_M\\ &\quad + C\left(R_2^{-1} + ||\rho||_{1+1/n}^{(n+1)/4}||\rho_2||_{\mathrm{pot}}\right)\\ &\leq Ch^{\mathrm{r}}_M M_1M_3 + C\left(R_2^{-1} + ||\rho||_{1+1/n}^{(n+1)/4}||\rho_2||_{\mathrm{pot}}\right).\end{aligned}$$

Here we used that for some constant $C > 0$ the following inequality holds:

$$x^{3/2} + y^{3/2} + z^{3/2} \leq 1 - C(xy + xz + yz) \text{ for } x,y,z \geq 0 \text{ with } x+y+z = 1.$$

Now we consider a minimizing sequence $(\rho_i) \subset \mathcal{F}_M^{\mathrm{r}}$ of $\mathcal{H}_{\mathcal{C}}^{\mathrm{r}}$ and choose shift vectors $(a_i) \subset \mathbb{R}^2$, $\delta_0 > 0$, and $R_0 > 0$ according to Cor. 3.8. Since all our functionals are invariant under spatial translations the sequence $T_{a_i}\rho_i = \rho_i(\cdot + a_i)$ is again minimizing and hence bounded in $L^{1+1/n}(\mathbb{R}^2)$ so that up to a subsequence we can assume that it converges weakly to some $\rho_0 \in L^{1+1/n}(\mathbb{R}^2)$. We choose $R_1 > R_0$ so that by Cor. 3.8, $M_{i,1} \geq \delta_0$ for i large, and

$$-Ch_M^{\mathrm{r}}\delta_0 M_{i,3} \leq C\,R_2^{-1} + C||\rho_{0,2}||_{\mathrm{pot}} + C||\rho_{i,2} - \rho_{0,2}||_{\mathrm{pot}} + \mathcal{H}_{\mathcal{C}}^{\mathrm{r}}(T_{a_i}\rho_i) - h_M^{\mathrm{r}}.$$

Given any $\varepsilon > 0$ we increase $R_1 > R_0$ such that the second term on the right hand side is smaller than ε. Next we choose $R_2 > 2R_1$ such that the first term is small. Now that R_1 and R_2 are fixed, the third term converges to zero by Lemma 3.9, and since $T_{a_i}\rho_i$ is minimizing the remainder follows suit. Therefore for i sufficiently large,

$$\int_{B_{R_2}} T_{a_i}\rho_i \,\mathrm{d}x = M - M_{i,3} \geq M - (-Ch_M^{\mathrm{r}}\delta_0)^{-1}\varepsilon.$$

The strong convergence of the potential energies now follows by Lemma 3.9. By weak convergence $\rho_0 \geq 0$ a.e., and for any $\varepsilon > 0$ there exists $R > 0$ such that

$$M \geq \int_{B_R} \rho_0 \,\mathrm{d}x \geq M - \varepsilon,$$

in particular $\rho_0 \in L^1(\mathbb{R}^2)$ with $\int \rho_0 = M$. The functional $\rho \mapsto \int \Psi(\rho)\,\mathrm{d}x$ is convex, so by Mazur's lemma [19, 2.13] and Fatou's lemma

$$\int \Psi(\rho_0)\,\mathrm{d}x \leq \limsup_{i\to\infty} \int \Psi(T_{a_i}\rho_i)\,\mathrm{d}x.$$

Hence $\rho_0 \in \mathcal{F}_M^{\mathrm{r}}$ with

$$\mathcal{H}_{\mathcal{C}}^{\mathrm{r}}(\rho_0) \leq \limsup_{i\to\infty} \mathcal{H}_{\mathcal{C}}^{\mathrm{r}}(\rho_i) = h_M^{\mathrm{r}},$$

and the proof is complete. □

Remark. (a) Thm. 3.10 provides a minimizer ρ_0 of the reduced energy-Casimir functional $\mathcal{H}_{\mathcal{C}}^{\mathrm{r}}$ under the assumptions (Ψ1)–(Ψ3). By Thm. 3.3 this minimizer can be lifted to a minimizer f_0 of the original energy-Casimir functional $\mathcal{H}_{\mathcal{C}}$. By Lemma 3.2 the function Ψ satisfies the necessary assumptions if Φ which appears in the original Casimir functional satisfies the following ones: $\Phi \in C^1([0,\infty[)$, $\Phi(0) = \Phi'(0) = 0$ and

(Φ1) Φ is strictly convex,

(Φ2) $\Phi(f) \geq Cf^{1+1/k}$ for $f \geq 0$ large,

(Φ3) $\Phi(f) \leq Cf^{1+1/k'}$ for $f \geq 0$ small, with growth rates $k, k' \in]0,1[$.

(b) As will be seen in the next section the mere fact that f_0 minimizes $\mathcal{H}_\mathcal{C}$ is not sufficient for stability. However, let $(f_i) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}_\mathcal{C}$. By Thm. 3.3 (a) the sequence of induced spatial densities $\rho_i = \rho_{f_i}$ is minimizing for $\mathcal{H}_\mathcal{C}^{\mathrm{r}}$. Choose a subsequence of (ρ_i) (and (f_i)) and shift vectors such that the assertions of Thm. 3.10 hold, and denote the shifted subsequence again by (f_i). We claim that this sequence converges weakly to f_0. Clearly, (f_i) is bounded in $L^{1+1/k}(\mathbb{R}^4)$ with bounded kinetic energy, and $E_{\mathrm{pot}}(f_i) = E_{\mathrm{pot}}(\rho_i) \to E_{\mathrm{pot}}(\rho_0)$. Any subsequence of (f_i) must therefore have a weakly convergent subsequence with weak limit $\tilde{f}_0$ which is a minimizer of $\mathcal{H}_\mathcal{C}$ and induces the same spatial density ρ_0 and potential U_0. But then by Thm. 3.3, $\tilde{f}_0 = f_0$ so that indeed $f_i \rightharpoonup f_0$ weakly in $L^{1+1/k}(\mathbb{R}^4)$.

(c) For $k \geq 1$ one can still obtain stability results, cf. [8] for the Kuzmin disk which corresponds to $\Phi(f) = f^{3/2}$, i.e., $k = k' = 2$. However, the reduction approach cannot work, because as we shall see in the last section this approach implies stability for the Euler-Poisson system where stability is probably lost at $n = 2$, i.e., $k = 1$.

3.3 Stability of minimizers

Now that the existence of a minimizer is proven, we can explore its dynamical stability properties. So let ρ_0 be as obtained in Thm. 3.10 and f_0 as induced by Thm. 3.3. A simple expansion shows that

$$\mathcal{H}_\mathcal{C}(f) - \mathcal{H}_\mathcal{C}(f_0) = d(f, f_0) + E_{\mathrm{pot}}(f - f_0), \tag{3.2}$$

where for $f \in \mathcal{F}_M$ and with the Lagrange multiplier E_0 from Thm. 3.3 (b),

$$\begin{aligned} d(f, f_0) &:= \iint [\Phi(f) - \Phi(f_0) + E(f - f_0)]\,\mathrm{d}v\,\mathrm{d}x \\ &= \iint [\Phi(f) - \Phi(f_0) + (E - E_0)(f - f_0)]\,\mathrm{d}v\,\mathrm{d}x \\ &\geq \iint [\Phi'(f_0) + (E - E_0)](f - f_0)\,\mathrm{d}v\,\mathrm{d}x \geq 0 \end{aligned}$$

with $d(f, f_0) = 0$ iff $f = f_0$. For the positivity of d we use the strict convexity of Φ and the form of f_0 according to Thm. 3.3 (b); by that theorem the term in brackets vanishes on the support of f_0. We recall that $-E_{\mathrm{pot}}(f) = \langle \rho_f, \rho_f \rangle_{\mathrm{pot}} = ||\rho_f||_{\mathrm{pot}}^2$ defines a norm on ρ_f, cf. Lemma 3.1; note that the right hand side in Eqn. (3.2) is $d(f, f_0) - ||\rho_f - \rho_0||_{\mathrm{pot}}^2$. We obtain the following stability result; $C_\mathrm{c}^2(\mathbb{R}^4)$ denotes the space of compactly supported C^2 functions on $\mathbb{R}^4$.

Theorem 3.11. *Let f_0 be a minimizer of $\mathcal{H}_\mathcal{C}$ on $\mathcal{F}_M$ obtained from a minimizer ρ_0 of $\mathcal{H}_\mathcal{C}^{\mathrm{r}}$, and assume that the minimizer is unique. Then for any $\varepsilon > 0$ there exists $\delta > 0$*

such that for any classical solution $[0,T[\ni t\mapsto f(t)$ *of the flat Vlasov-Poisson system with* $f(0)\in C^2_{\mathrm{c}}(\mathbb{R}^4)\cap\mathcal{F}_M$ *and* $||f(0)||_{1+1/k}=||f_0||_{1+1/k}$ *the estimate*

$$d(f(0),f_0)+||\rho_{f(0)}-\rho_0||^2_{\mathrm{pot}}<\delta$$

implies that for each $t\in[0,T[$ *there exists a shift vector* $a\in\mathbb{R}^2$ *such that*

$$||f(t)-T_af_0||_{1+1/k}+d(f(t),T_af_0)+||\rho_{f(t)}-T_a\rho_0||^2_{\mathrm{pot}}<\varepsilon.$$

Proof. Assume that the assertion were false. Then there exists $\varepsilon>0$, $t_j>0$, $f_j(0)\in C^2_{\mathrm{c}}(\mathbb{R}^4)\cap\mathcal{F}_M$ with $||f_j(0)||_{1+1/k}=||f_0||_{1+1/k}$ such that for every $j\in\mathbb{N}$,

$$d(f_j(0),f_0)-E_{\mathrm{pot}}(f_j(0)-f_0)<\frac{1}{j} \tag{3.3}$$

but for any shift vector $a\in\mathbb{R}^2$,

$$||f_j(t_j)-T_af_0||_{1+1/k}+d(f_j(t_j),T_af_0)-E_{\mathrm{pot}}(f_j(t_j)-T_af_0)\geq\varepsilon. \tag{3.4}$$

Since $\mathcal{H}_\mathcal{C}$ is preserved along solutions we have from (3.2) and (3.3) that

$$\mathcal{H}_\mathcal{C}(f_j(t_j))=\mathcal{H}_\mathcal{C}(f_j(0))\to\mathcal{H}_\mathcal{C}(f_0),$$

i.e., $(f_j(t_j))$ is a minimizing sequence. By Thm. 3.10 and the remark at the end of the previous section there is a sequence of shift vectors $(a_j)\subset\mathbb{R}^2$ such that up to a subsequence,

$$\lim_{j\to\infty}E_{\mathrm{pot}}(T_{a_j}f_j(t_j)-f_0)\to 0.$$

By (3.2) this implies that $d(T_{a_j}f_j(t_j),f_0)\to 0$. For the convergence of $||\cdot||_{1+1/k}$ we use the fact that $||f_j(t)||_{1+1/k}=||f_j(0)||_{1+1/k}=||f_0||_{1+1/k}$ for any $t>0$. By the remark, $T_{a_j}f_j(t_j)\rightharpoonup f_0$ weakly in $L^{1+1/k}(\mathbb{R}^4)$, and hence $T_{a_j}f_j(t_j)\to f_0$ strongly in $L^{1+1/k}(\mathbb{R}^4)$. But these convergence results for $T_{a_j}f_j(t_j)$ contradict (3.4). □

Remark. (a) The uniqueness assumption on f_0 in the above theorem is made mostly in order to avoid technical complications. It suffices if f_0 is isolated with respect to the topology of our stability estimate. If there should be a continuum of minimizers then the set of minimizers itself is stable; we refer to [16, Thm. 4] for such a formulation of the result in the three dimensional case. We are not aware of a case where there is a continuum of minimizers with fixed mass M. For a closely related variational problem it has been shown that the above stability estimate remains valid even then [30].

(b) As opposed to the three dimensional case [20, 27, 29] there is no global existence and uniqueness result to the initial value problem for the flat Vlasov-Poisson system yet. Hence our stability result is conditional in the sense that it holds as long as a suitable solution exists. A local existence and uniqueness result for smooth solutions with initial data in $C^2_{\mathrm{c}}(\mathbb{R}^4)$ as well as a global existence result for weak solutions to the flat system was established in [5]. We could also carry out our stability analysis in the framework of these global weak solutions, but this would only bury the main ideas under technicalities.

(c) By interpolation between L^1 and $L^{1+1/k}$ we obtain a stability estimate for $||f(t) - T_a f_0||_p$ with $p \in]1, 1+1/k]$. If we assume that the initial perturbations have supports of uniformly bounded measure we can include the case $p = 1$, if we assume a uniform bound on the L^∞ norm of the initial perturbations we can by interpolation include all $p \in [1+1/k, \infty[$; notice that both the measure of the support and the L^∞ norm are invariant under classical solutions of the Vlasov-Poisson system.

(d) The need for the shifts in the stability estimate arises from the Galilei invariance of the Vlasov-Poisson system. If f_0 is a steady state then for any fixed $V \in \mathbb{R}^2$ the function $f_0(x - tV, v - V)$ is a time dependent solution; f_0 is simply put into a uniformly moving coordinate system. But while the distance of this perturbation to the steady state grows linearly in t, it is arbitrarily close to the steady state at $t = 0$ for V small.

3.4 Connection to the Euler–Poisson system

A self-gravitating matter distribution can be described on the microscopic, kinetic level represented by the Vlasov-Poisson system or on the macroscopic, fluid level represented by the Euler-Poisson system. The reduction technique connects the stability problems for these two viewpoints. In the three dimensional situation this connection was observed in [25]. In the flat case the corresponding Euler-Poisson system reads

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) = 0,$$

$$\rho \frac{\partial u}{\partial t} + (u \cdot \nabla_x) u = -\nabla_x p - \rho \nabla_x U,$$

$$U(t,x) = -\int \frac{\rho(t,y)}{|x-y|}\,\mathrm{d}y,$$

with the equation of state

$$p(\rho) = \rho \Psi'(\rho) - \Psi(\rho).$$

Here p denotes the pressure of the fluid and u denotes its velocity field; the meaning of ρ and U is as before. If ρ_0 is a minimizer of the reduced energy-Casimir functional $\mathcal{H}^{\mathrm{r}}_{\mathcal{C}}$, then using the Euler-Lagrange identity in Thm. 3.3 (b) it is easy to check that ρ_0 and the zero velocity field $u_0 \equiv 0$ solve the flat Euler-Poisson system. Clearly, the state (ρ_0, u_0) minimizes the energy

$$\mathcal{H}(\rho, u) = \frac{1}{2} \int |u|^2 \rho \,\mathrm{d}x + \int \Psi(\rho)\,\mathrm{d}x + E_{\mathrm{pot}}(\rho),$$

among the states with $\int \rho = M$. Formally, the energy is conserved along solutions of the Euler-Poisson system. An expansion about (ρ_0, u_0) gives

$$\mathcal{H}(\rho, u) - \mathcal{H}(\rho_0, u_0) = \frac{1}{2} \int |u|^2 \rho \, \mathrm{d}x + d(\rho, \rho_0) + E_{\mathrm{pot}}(\rho - \rho_0),$$

where

$$d(\rho, \rho_0) := \int [\Psi(\rho) - \Psi(\rho_0) + (U_0 - E_0)(\rho - \rho_0)] \, \mathrm{d}x \geq 0.$$

Now the stability proof proceeds in the same way as in the Vlasov case. We can for every $\varepsilon > 0$ find a $\delta > 0$ such that for every solution $t \mapsto (\rho(t), u(t))$ of the flat Euler-Poisson system with $\rho(0) \in \mathcal{F}_M^{\mathrm{r}}$, which preserves energy and mass, the initial estimate

$$\frac{1}{2} \int |u(0)|^2 \rho(0) \, \mathrm{d}x + d(\rho(0), \rho_0) + ||\rho(0) - \rho_0||_{\mathrm{pot}}^2 < \delta$$

implies that as long as the solution exists and up to shifts in space,

$$||\rho(t) - \rho_0||_{1+1/n} + \frac{1}{2} \int |u(t)|^2 \rho(t) \, \mathrm{d}x + d(\rho(t), \rho_0) + ||\rho(t) - \rho_0||_{\mathrm{pot}}^2 < \varepsilon.$$

Neither in the flat case nor in the three dimensional one is there an existence theory for global solutions of the Euler-Poisson system, which preserve all the necessary quantities, so the result is conditional in this sense.

4 The Kuzmin disk

We will construct the Kuzmin disk by minimization of the energy functional $\mathcal{H}$ which is defined as

$$\begin{aligned}\mathcal{H}(f) &= \frac{1}{2}\iint |v|^2 f(x,v)\mathrm{d}x\mathrm{d}v - \frac{1}{2}\iint \frac{\rho_f(x)\rho_f(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \\ &=: E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f)\end{aligned} \tag{4.1}$$

over the set

$$\mathcal{F}_M := \left\{ f \in L^1_+(\mathbb{R}^4); \iint f^{3/2}\mathrm{d}x\mathrm{d}v = M, E_{\mathrm{kin}}(f) < \infty \right\}, \tag{4.2}$$

where $M > 0$ is an arbitrary prescribed constant. The technical difficulties of the limiting case arise from the fact that the kinetic energy, potential energy as well as the norm $||\cdot||_{3/2}$ are invariant under the transformation

$$S_\lambda f := \lambda^{-4} f(\lambda^{-4}x, \lambda v), \quad \lambda > 0. \tag{4.3}$$

This kind of scaling invariance is not present in the cases $k < 2$.

We will proceed as follows. In Section 4.1 we derive the explicit formula for the Kuzmin disk by minimizing the energy functional and using the sharp constant in the Hardy-Littlewood-Sobolev inequality, in analogy to [2], where the optimal Sobolev inequality was used to construct the Plummer sphere. Since we are dealing with an infinite-dimensional dynamical system, the fact that this state minimizes the energy does not automatically imply its stability. We will later see that the crucial property for the stability to hold (at least in our variational approach) is convergence of the potential energies along minimizing sequences of $\mathcal{H}$. In Section 4.2 we prove this convergence as well as the stability result with respect to perturbations restricted to the plane in which the steady state lives.

4.1 Existence

The main result of this section is the following theorem:

Theorem 4.1. *There exists a unique minimizer f_0 of $\mathcal{H}$ on $\mathcal{F}_M$ which, up to translations in x-space and scalings (4.3), has the form*

$$f_0(x,v) = [(-C_M E(x,v))_+]^2 \qquad a.e. \tag{4.4}$$

with potential U_0 and density ρ_0 given by (again up to translations and scalings corresponding to (4.3),

$$\rho_{f_0}(x) = C_M \left(\frac{1}{1+|x|^2}\right)^{3/2}, \qquad U_{f_0}(x) = -C_M \left(\frac{1}{1+|x|^2}\right)^{1/2}.$$

f_0 is indeed a steady-state of (2.2).

Proof. For each $f \in \mathcal{F}_M$ we define

$$h_f(x,v) := \frac{1}{2}|v|^2 + U_f(x),$$

$$N_f^{-/+} := \{(x,v) \in \mathbb{R}^4; h_f(x,v) < 0/ \geq 0\}.$$

Then the energy functional can be written as

$$\mathcal{H}(f) = \int_{N_f^-} h_f f \mathrm{d}x\mathrm{d}v + \int_{N_f^+} h_f f \mathrm{d}x\mathrm{d}v + |E_{\text{pot}}(f)|. \tag{4.5}$$

We can use Hölder's inequality to estimate the first term on the right hand side as follows:

$$-\int_{N_f^-} h_f f \mathrm{d}x\mathrm{d}v \leq \left[\int f^{3/2} \mathrm{d}x\mathrm{d}v\right]^{2/3} \left[\int_{N_f^-} |h_f|^3 \mathrm{d}x\mathrm{d}v\right]^{1/3}. \tag{4.6}$$

A simple computation gives us the value of the last integral:

$$\begin{aligned}
\int_{N_f^-} |h_f|^3 \mathrm{d}x\mathrm{d}v &= 2\pi \int_{\mathbb{R}^2} \int_0^{\sqrt{-2U_f(x)}} \left|\frac{1}{2}|v|^2 + U_f(x)\right|^3 |v| \mathrm{d}|v| \mathrm{d}x \\
&= 2\pi \int_{\mathbb{R}^2} \int_0^{U_f(x)} s^3 \mathrm{d}s \mathrm{d}x = \frac{\pi}{2} \int_{\mathbb{R}^2} |U_f(x)|^4 \mathrm{d}x = \frac{\pi}{2} ||U_f||_4^4.
\end{aligned}$$

In [2] the Sobolev inequality was used to estimate the L^p norm of the potential by the norm of ∇U. This is unfortunately not possible here since we have no information about the gradient of the potential. Instead we use the Hölder and the Hardy-Littlewood-Sobolev inequality to achieve a similar estimate. By the reflexivity of $L^4(\mathbb{R}^2)$ there exists $\varphi \in L^{4/3}(\mathbb{R}^2)$ with $||\varphi||_{4/3} = 1$ such that the following holds:

$$\begin{aligned}
||U_f||_4^4 &= \left[\int U_f \varphi \mathrm{d}x\right]^4 = 16 \langle \rho_f, \varphi \rangle_{\text{pot}}^4 \leq 16 ||\rho_f||_{\text{pot}}^4 ||\varphi||_{\text{pot}}^4 && (4.7) \\
&= 4|E_{\text{pot}}(f)|^2 \left[\iint \frac{\varphi(x)\varphi(y)}{|x-y|} \mathrm{d}x\mathrm{d}y\right]^2 \leq 4C_{\text{HLS}}^2 |E_{\text{pot}}(f)|^2 ||\varphi||_{4/3}^4 && (4.8) \\
&= 4C_{\text{HLS}}^2 |E_{\text{pot}}(f)|^2,
\end{aligned}$$

where we recall the definitions of $\langle\cdot,\cdot\rangle_{\mathrm{pot}}$ and $||\cdot||_{\mathrm{pot}}$ as introduced in [7]:

$$\begin{aligned}
\langle\xi,\psi\rangle_{\mathrm{pot}} &:= \frac{1}{2}\iiiint \frac{\xi(x,v)\psi(y,w)}{|x-y|}\mathrm{d}x\mathrm{d}y\mathrm{d}v\mathrm{d}w \\
&= \frac{1}{2}\iint \frac{\rho_\xi(x)\rho_\psi(y)}{|x-y|}\mathrm{d}x\mathrm{d}y =: \langle\rho_\xi,\rho_\psi\rangle_{\mathrm{pot}}, \\
||\xi||_{\mathrm{pot}} &:= \langle\xi,\xi\rangle_{\mathrm{pot}}^{1/2} = \sqrt{-E_{\mathrm{pot}}(\xi)} \\
&= \sqrt{-E_{\mathrm{pot}}(\rho_\xi)} =: ||\rho_\xi||_{\mathrm{pot}}, \\
||\xi||_{\mathrm{pot}} &\le C||\rho_\xi||_{4/3} \le C||\xi||_{3/2},
\end{aligned}$$

where $\xi,\psi\in L^{3/2}(\mathbb{R}^4)$. As one can see the above definitions are equivalent for distribution functions as well as for their induced spatial densities (by integrating over velocities). In this paper we will use both notations interchangeably. The above expressions define a scalar product and induced norm on $L^{3/2}(\mathbb{R}^4)$ (resp. $L^{4/3}(\mathbb{R}^2)$). Now from (4.5) we have

$$\begin{aligned}
\mathcal{H}(f) &\ge \int_{N_f^+} h_f f\mathrm{d}x\mathrm{d}v + |E_{\mathrm{pot}}(f)| - \left(2\pi M^2 C_{\mathrm{HLS}}^2\right)^{1/3}|E_{\mathrm{pot}}(f)|^{2/3} \\
&\ge -\frac{8\pi}{27}M^2 C_{\mathrm{HLS}}^2, \qquad (4.9)
\end{aligned}$$

where we have dropped the non-negative integral over N_f^+ and minimized with respect to $|E_{\mathrm{pot}}(f)|$ to obtain the last estimate. Now we investigate the optimal case when the inequalities in (4.6), (4.7) and (4.8) reduce to equalities. In (4.6) equality is achieved when

$$f(x,v) = \alpha|h_f(x,v)|^2\theta(-h_f(x,v)),$$

where α is some constant and θ is the usual Heaviside function. This implies that the density must be in the form

$$\rho_f(x) = \frac{2\pi}{3}\alpha|U_f(x)|^3.$$

The function φ in (4.7) must of course satisfy

$$|U_f(x)|^4 = \beta|\varphi(x)|^{4/3},$$

and finally equality in (4.8) holds when

$$\varphi(x) = \pi^{-3/4}\sigma^{-3/2}\left(1+\left|\frac{x-\xi}{\sigma}\right|^2\right)^{-3/2}, \qquad \sigma>0,\ \xi\in\mathbb{R}^2,$$

see [18]. With a little algebra we can combine the above four equations and show that α and β depend only on M with

$$\begin{aligned}
\alpha &= 2^{-6}3^4\pi^{-2}C_{\mathrm{HLS}}^{-4}M^{-2}, \\
\beta &= 2^{-2}3^2\pi^{-2}C_{\mathrm{HLS}}^{-2}\alpha^{-2},
\end{aligned}$$

and that energy of this optimal state is precisely the value which appeared as a lower bound in (4.9). The explicit forms of the density and potential of the minimizer are now

$$\rho_f(x) = C_M \sigma^{-3/2} \left(1 + \left|\frac{x-\xi}{\sigma}\right|^2\right)^{-3/2},$$

$$U_f(x) = -C_M \sigma^{-1/2} \left(1 + \left|\frac{x-\xi}{\sigma}\right|^2\right)^{-1/2}.$$

To verify (2.2b) we define for $\psi \in L^{4/3}(\mathbb{R}^2)$ functional

$$R(\varepsilon) := \frac{\langle \rho_f + \varepsilon\psi, \rho_f \rangle_{\text{pot}}}{||\rho_f + \varepsilon\psi||_{4/3} ||\rho_f||_{4/3}}$$

and since R has its maximum at $\varepsilon = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R(\varepsilon)\bigg|_{\varepsilon=0} = 0.$$

After some algebra we get (2.2b). □

4.2 Existence and stability

First we formulate some a-priori estimates.

Lemma 4.2. *For all $f \in \mathcal{F}_M$ the following holds:*

(a) $\rho_f \in L^{4/3}(\mathbb{R}^2)$ *with*

$$\int \rho_f^{4/3} \mathrm{d}x \leq \left(\iint f^{3/2} \mathrm{d}x\mathrm{d}v\right)^{2/3} \left(\iint |v|^2 f \mathrm{d}x\mathrm{d}v\right)^{1/3} \leq C E_{\text{kin}}(f)^{1/3}.$$

(b) $U_f \in L^4(\mathbb{R}^2)$ *with*

$$-E_{\text{pot}}(f) \leq C ||\rho_f||_{4/3}^2 \leq C E_{\text{kin}}(f)^{1/2}.$$

Proof. By splitting the v-integral according to $|v| \leq R$ and $|v| > R$ we get:

$$\begin{aligned} \rho_f(x) &= \int_{|v|\leq R} f(x,v)\mathrm{d}v + \int_{|v|>R} f(x,v)\mathrm{d}v \\ &\leq \pi^{1/3} \left(\int f^{3/2} \mathrm{d}v\right)^{2/3} R^{2/3} + R^{-2} \int |v|^2 f \mathrm{d}v =: I(R). \end{aligned}$$

Now we optimize $I(R)$ with respect to R:

$$\frac{\mathrm{d}I}{\mathrm{d}R} = \frac{2\pi^{1/3}}{3}\left[\int f^{3/2}\mathrm{d}v\right]^{2/3} R^{-1/3} - 2R^{-3}\int |v|^2 f\mathrm{d}v.$$

If we now set the derivative above equal to zero to obtain the minimum we find

$$R = \left[\frac{3\int |v|^2 f\mathrm{d}v}{\pi^{1/3}\left(\int f^{3/2}\right)^{2/3}}\right]^{3/8},$$

and

$$\rho_f(x) \leq C\left(\int f^{3/2}\mathrm{d}v\right)^{1/2}\left(\int |v|^2 f\mathrm{d}v\right)^{1/4}.$$

Hence by Hölder's inequality,

$$\begin{aligned}\int \rho_f^{4/3}\mathrm{d}x &\leq C\int\left(\int f^{3/2}\mathrm{d}v\right)^{2/3}\left(\int |v|^2 f\mathrm{d}v\right)^{1/3}\mathrm{d}x \\ &\leq C\left(\iint f^{3/2}\mathrm{d}x\mathrm{d}v\right)^{2/3}\left(\iint |v|^2 f\mathrm{d}x\mathrm{d}v\right)^{1/3} \\ &\leq CE_{\mathrm{kin}}(f)^{1/3},\end{aligned} \tag{4.10}$$

and the estimate (a) is proved. The assertion (b) follows by the Hardy-Littlewood-Sobolev inequality [19, Sec. 4.3]. □

A lower bound for $\mathcal{H}$ immediately follows from Lemma 4.2:

Corollary 4.3. *There exists a constant $C > 0$ such that for $f \in \mathcal{F}_M$*

$$\mathcal{H}(f) \geq E_{\mathrm{kin}}(f) - CE_{\mathrm{kin}}(f)^{1/2},$$

in particular,

$$h_M := \inf_{\mathcal{F}_M}\mathcal{H} > -\infty$$

and E_{kin} is bounded along minimizing sequences for $\mathcal{H}$ in $\mathcal{F}_M$.

Now we explore the behavior of $\mathcal{H}$ and $||f||_{3/2}$ under scaling transformations.

Lemma 4.4.

(a) *Let $M > 0$. Then $h_M < 0$.*

(b) *For all $M, \bar{M} > 0$,*

$$h_{\bar{M}} = \left(\frac{\bar{M}}{M}\right)^2 h_M. \tag{4.11}$$

Proof. For a given function $f(x,v)$ we define a rescaled function $\bar{f}(x,v) := f(ax,bv)$. Then

$$\iint \bar{f}^{3/2}\mathrm{d}x\mathrm{d}v = \iint (f(ax,bv))^{3/2}\mathrm{d}x\mathrm{d}v = (ab)^{-2}\iint f^{3/2}\mathrm{d}x\mathrm{d}v,$$

i.e., $f \in \mathcal{F}_M$ iff $\bar{f} \in \mathcal{F}_{\bar{M}}$ where $\bar{M} := (ab)^{-2}M$. Next

$$E_{\mathrm{kin}}(\bar{f}) = \frac{1}{2}\iint |v|^2 f(ax,bv)\mathrm{d}x\mathrm{d}v = a^{-2}b^{-4}E_{\mathrm{kin}}(f),$$

$$E_{\mathrm{pot}}(\bar{f}) = -\frac{1}{2}\iiiint \frac{f(ax,bv)f(ay,bw)}{|x-y|}\mathrm{d}w\mathrm{d}v\mathrm{d}x\mathrm{d}y = a^{-3}b^{-4}E_{\mathrm{pot}}(f).$$

To prove (a) we fix any $f \in \mathcal{F}_M$ and let $a = b^{-1}$ so that $\bar{f} \in \mathcal{F}_M$ as well. Then

$$\mathcal{H}(\bar{f}) = b^{-2}E_{\mathrm{kin}}(f) + b^{-1}E_{\mathrm{pot}}(f) < 0$$

for $b > 0$ sufficiently large, since $E_{\mathrm{pot}}(f) < 0$. To prove (b) we choose $a = 1$. Then

$$\mathcal{H}(\bar{f}) = b^{-4}\mathcal{H}(f), \tag{4.12}$$

and since $b^{-2} = \left(\bar{M}/M\right)$ and the mapping $f \mapsto \bar{f}$ is one-to-one and onto from $\mathcal{F}_M$ to $\mathcal{F}_{\bar{M}}$, (b) follows. □

The key step in our stability analysis is the convergence of the potential energy along minimizing sequences. This will be treated by the following series of lemmas:

Lemma 4.5. *If (g_i) is a sequence which converges weakly to g in $L^1(\mathbb{R}^2)$ and (h_i) is a sequence bounded in $L^\infty(\mathbb{R}^2)$ which converges almost everywhere to h, then*

$$g_i h_i \rightharpoonup gh \text{ weakly in } L^1(\mathbb{R}^2).$$

For a proof we refer to [10, Prop. 5]. In order to analyze the convergence of the potential energy along the minimizing sequence we first consider axially symmetric densities.

Definition 4.6. *A function $f : \mathbb{R}^4 \mapsto \mathbb{R}$ is called axially symmetric if*

$$f(x,v) = f(Ax,Av), \qquad x,v \in \mathbb{R}^2, A \in \mathrm{SO}(2).$$

A function $\rho : \mathbb{R}^2 \mapsto \mathbb{R}$ is called axially symmetric if

$$\rho(x) = \rho(Ax), \qquad x \in \mathbb{R}^2, A \in \mathrm{SO}(2).$$

Later we will remove the symmetry restriction. Note that when viewed as a function on $\mathbb{R}^6$ respectively $\mathbb{R}^3$ through

$$\begin{aligned}\tilde{f}(x_1,x_2,x_3,v_1,v_2,v_3) &= f(x_1,x_2,v_1,v_2)\delta(x_3)\delta(v_3),\\ \tilde{\rho}(x_1,x_2,x_3) &= \rho(x_1,x_2)\delta(x_3),\end{aligned}$$

these functions are really axially symmetric with respect to the x_3-axis.

Lemma 4.7. *Let* $(\rho_i) \subset L^{4/3}(\mathbb{R}^2)$ *be bounded, axially symmetric,* supp $\rho_i \subset B_{R_1,R_2}$, *where* $B_{R_1,R_2} := \{x \in \mathbb{R}^2; 0 < R_1 < |x| < R_2 < \infty\}$, *and assume that*

$$\rho_i \rightharpoonup \rho_0 \text{ weakly in } L^{4/3}(\mathbb{R}^2).$$

Then

$$E_{\text{pot}}(\rho_i - \rho_0) \to 0.$$

Proof. First we note that due to the condition of the support of (ρ_i) this sequence is also bounded in $L^1(\mathbb{R}^2)$ and has the property

$$\rho_i \rightharpoonup \rho_0 \text{ weakly in } L^1 \cap L^{4/3}(\mathbb{R}^2).$$

We would like to apply Lemma 4.5 to the sequences (ρ_i) and $(U_i) := (U_{\rho_i})$. The boundedness of (U_i) in $L^\infty(\mathbb{R}^2)$ can be proved using estimates from [21] for the symmetric potentials:

$$\begin{aligned}
|U_i(x)| &= |U_i(r)| = 4\left|\int_{R_1}^{R_2} \frac{s}{r+s}\rho_i(s) K\left(\frac{2\sqrt{rs}}{r+s}\right) \mathrm{d}s\right| \\
&\leq C\int_{R_1}^{R_2} s\rho_i(s) K\left(\frac{2\sqrt{rs}}{r+s}\right) \mathrm{d}s \\
&\leq C||\rho_i||_{4/3}\left(\int_{R_1}^{R_2} s\left[K\left(\frac{2\sqrt{rs}}{r+s}\right)\right]^4 \mathrm{d}s\right)^{1/4} \\
&\leq C + C\left(\int_{R_1}^{R_2} |\ln(1-[r,s])|^4 \mathrm{d}s\right)^{1/4},
\end{aligned}$$

where $r := |x|$, $[r,s] := \min\{r/s, s/r\}$ and K denotes complete elliptic integral of the first kind defined as

$$K(\xi) := \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-\xi^2\sin^2\varphi}} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-\xi^2 t^2}\sqrt{1-t^2}},\ 0 \leq \xi < 1.$$

Now we can estimate the logarithm as

$$|\ln(1-[r,s])| \leq \begin{cases} C\left(1-\frac{r}{s}\right)^{-1/8} & 0 \leq r \leq s, \\ C\left(1-\frac{s}{r}\right)^{-1/8} & 0 \leq s \leq r. \end{cases} \tag{4.13}$$

With this estimate we can according to the position of r estimate the value of the corresponding potential. In the case $0 \leq r \leq R_1$ we get

$$\int_{R_1}^{R_2} |\ln(1-[r,s])|^4 \mathrm{d}s \leq \int_{R_1}^{R_2} \left(1-\frac{r}{s}\right)^{-1/2} \mathrm{d}s \leq C\int_{R_1}^{R_2} (s-R_1)^{-1/2} \mathrm{d}s < \infty,$$

for $R_1 \leq r \leq R_2$

$$\begin{aligned}\int_{R_1}^{R_2} |\ln(1-[r,s])|^4 \mathrm{d}s &\leq C\left(\int_{R_1}^{r}\left(1-\frac{s}{r}\right)^{-1/2}\mathrm{d}s + \int_{r}^{R_2}\left(1-\frac{r}{s}\right)^{-1/2}\mathrm{d}s\right)\\ &\leq C(1+4(R_2-R_1)^{1/2}) < \infty,\end{aligned}$$

and for $r \geq R_2$

$$\int_{R_1}^{R_2} |\ln(1-[r,s])|^4 \mathrm{d}s \leq C R_1^{1/2}(R_2-R_1)^{1/2} < \infty.$$

Now

$$\begin{aligned}U_i(r) - U_0(r) &= -4\int_{R_1}^{R_2} \frac{s}{r+s}(\rho_i(s)-\rho_0(s)) K\left(\frac{2\sqrt{rs}}{r+s}\right)\mathrm{d}s\\ &= -4\int_{R_1}^{R_2} s(\rho_i(s)-\rho_0(s)) K\left(\frac{2\sqrt{rs}}{r+s}\right)\frac{1}{r+s}\mathrm{d}s \to 0 \text{ a.e. } i\to\infty,\end{aligned}$$

since for each $r > 0$,

$$K\left(\frac{2\sqrt{r\cdot}}{r+\cdot}\right) \in L^4([R_1,R_2]).$$

The latter follow easily from (4.13). Now we can use Lemma 4.5 to conclude that

$$E_{\mathrm{pot}}(\rho_i) = \frac{1}{2}\int U_i\rho_i \mathrm{d}x \to \frac{1}{2}\int U_0\rho_0\mathrm{d}x = E_{\mathrm{pot}}(\rho_0).$$

Now

$$E_{\mathrm{pot}}(\rho_i-\rho_0) = E_{\mathrm{pot}}(\rho_i) - E_{\mathrm{pot}}(\rho_0) - \int U_0(\rho_i-\rho_0),$$

and the fact that $U_0 \in L^4(\mathbb{R}^2)$ and the weak convergence of (ρ_i) in $L^{4/3}(\mathbb{R}^2)$ prove the assertion. □

Now that the convergence of the potential energies is proved, the existence of a minimizer follows. The restriction that we obtain a minimizer in the class of axially symmetric functions in $\mathcal{F}_M$ will be removed below.

Theorem 4.8. *Let (f_i) be a minimizing sequence of $\mathcal{H}$ in $\mathcal{F}_M$ which is axially symmetric. Then there exists $f_0 \in \mathcal{F}_M$, $(\lambda_i) \subset \mathbb{R}^+$ and a subsequence of (f_i) (also denoted by (f_i)) such that*

$$S_{\lambda_i} f_i \rightharpoonup f_0 \text{ weakly in } L^{3/2}(\mathbb{R}^4),$$

$$E_{\mathrm{pot}}(S_{\lambda_i} f_i - f_0) \to 0.$$

Moreover, f_0 is a minimizer of $\mathcal{H}$ over $\mathcal{F}_M^{\mathrm{S}} := \{f \in \mathcal{F}_M; f \text{ axially symmetric}\}$.

Proof. Let (f_i) be a minimizing sequence. From the definition of $\mathcal{F}_M$ we have

$$||f_i||_{3/2}^{3/2} = M.$$

First we choose λ_i, such that the sequence $(S_{\lambda_i} f_i)$ has the following property

$$\int_{|x|<1} \int (S_{\lambda_i} f_i)^{3/2} \mathrm{d}v\mathrm{d}x = \int_{|x|\geq 1} \int (S_{\lambda_i} f_i)^{3/2} \mathrm{d}v\mathrm{d}x = M/2, \quad i \in \mathbb{N}. \tag{4.14}$$

For notational simplicity we will from now one denote by (f_i) the sequence $(S_{\lambda_i} f_i)$ which has the property (4.14). Because of the boundedness of (f_i) in $L^{3/2}(\mathbb{R}^4)$ we can extract a subsequence (without change of notation) such that

$$f_i \rightharpoonup f_0 \text{ weakly in } L^{3/2}(\mathbb{R}^4).$$

Clearly, f_0 is non-negative and axially symmetric. From the weak convergence we have

$$E_{\mathrm{kin}}(f_0) \leq \liminf_{i\to\infty} E_{\mathrm{kin}}(f_i) < \infty, \tag{4.15}$$

$$\iint f_0^{3/2} \leq \liminf_{i\to\infty} \iint f_i^{3/2} = M. \tag{4.16}$$

We have to show that f_0 is actually a minimizer of $\mathcal{H}$. The key step is to show that $\iint f_i^{3/2}$ cannot concentrate either at the origin or at infinity. We claim that for each $\varepsilon > 0$ there exists $N \geq 1$ such that

$$\limsup_{i\to\infty} \int_{r\leq 1/N} \int f_i^{3/2} \mathrm{d}v\mathrm{d}x + \limsup_{i\to\infty} \int_{r\geq N} \int f_i^{3/2} \mathrm{d}v\mathrm{d}x < \varepsilon. \tag{4.17}$$

If not, there exists an $\varepsilon_0 > 0$ such that up to a subsequence

$$\int_{r\leq 1/N} \int f_i^{3/2} \mathrm{d}v\mathrm{d}x + \int_{r\geq N} \int f_i^{3/2} \mathrm{d}v\mathrm{d}x \geq \varepsilon_0 \tag{4.18}$$

for any $N \geq 1$. Since $f_0 \in L^{3/2}(\mathbb{R}^4)$, for any $\varepsilon > 0$ there exists $N_0 \geq 2$ such that

$$\int_{r\leq 1/N_0} \int f_0^{3/2} \mathrm{d}v\mathrm{d}x + \int_{r\geq N_0} \int f_0^{3/2} \mathrm{d}v\mathrm{d}x < \varepsilon. \tag{4.19}$$

We now split f_i into four parts. The essential part f_i^{e} is defined as

$$f_i^{\mathrm{e}} := f_i \mathbf{1}_{1/N_0 \leq r \leq N_0}.$$

The small part f_i^{s} is defined as

$$f_i^{\mathrm{s}} := f_i [\mathbf{1}_{1/N_0^4 \leq r \leq 1/N_0} + \mathbf{1}_{N_0 \leq r \leq N_0^4}].$$

And finally the two concentrated parts are defined as

$$f_i^{\mathrm{c}} = f_i^{\mathrm{c}_1} + f_i^{\mathrm{c}_2} := f_i \mathbf{1}_{r \le 1/N_0^4} + f_i \mathbf{1}_{r \ge N_0^4}.$$

The energy split corresponding to the parts just described is

$$\begin{aligned} \mathcal{H}(f_i) \quad = \quad & \mathcal{H}(f_i^{\mathrm{e}}) + \mathcal{H}(f_i^{\mathrm{s}}) + \mathcal{H}(f_i^{\mathrm{c}_1}) + \mathcal{H}(f_i^{\mathrm{c}_2}) \\ & -2(\langle f_i^{\mathrm{s}}, f_i^{\mathrm{e}} + f_i^{\mathrm{c}} \rangle_{\mathrm{pot}} + \langle f_i^{\mathrm{c}_1}, f_i^{\mathrm{e}} \rangle_{\mathrm{pot}} + \langle f_i^{\mathrm{c}_1} + f_i^{\mathrm{e}}, f_i^{\mathrm{c}_2} \rangle_{\mathrm{pot}}). \end{aligned} \tag{4.20}$$

The crucial step is to estimate the interaction terms in (4.20). The first term is easily bounded by

$$\langle f_i^{\mathrm{s}}, f_i^{\mathrm{e}} + f_i^{\mathrm{c}} \rangle_{\mathrm{pot}} \le ||f_i^{\mathrm{s}}||_{\mathrm{pot}} ||f_i||_{\mathrm{pot}} \le C ||f_i^{\mathrm{s}}||_{\mathrm{pot}}$$

By Lemma 4.7, (4.10) and (4.19), we have for fixed N_0 (up to a subsequence)

$$\begin{aligned} ||f_i^{\mathrm{s}}||_{\mathrm{pot}} \to ||f_0^{\mathrm{s}}||_{\mathrm{pot}} \quad & \le \quad C ||\rho_{f_0^{\mathrm{s}}}||_{4/3} \\ & \le \quad C \left\{ \iint \left| f_0 \left[\mathbf{1}_{r \le 1/N_0} + \mathbf{1}_{r \ge N_0} \right] \right|^{3/2} \right\}^{1/2} < C \varepsilon^{1/2}. \end{aligned}$$

The estimate for the second term follows:

$$\begin{aligned} 2 \langle f_i^{\mathrm{c}_1}, f_i^{\mathrm{e}} \rangle_{\mathrm{pot}} \quad & = \quad \iint \frac{\rho_i^{\mathrm{c}_1}(x) \rho_i^{\mathrm{e}}(y)}{|x-y|} \mathrm{d}x \mathrm{d}y \\ & \le \quad \left(\frac{1}{N_0} - \frac{1}{N_0^4} \right)^{-1} \int_{B_{1/N_0^4}} \rho_i(x) \mathrm{d}x \int_{B_{N_0}} \rho_i(x) \mathrm{d}x \\ & \le \quad C \left(\frac{1}{N_0} - \frac{1}{N_0^4} \right)^{-1} ||\rho_i||_{4/3}^2 \frac{1}{N_0^2} N_0^{1/2} \le C N_0^{-1/2}. \end{aligned}$$

The last term can be treated as

$$\begin{aligned} \langle f_i^{\mathrm{e}} + f_i^{\mathrm{c}_1}, f_i^{\mathrm{c}_2} \rangle_{\mathrm{pot}} \quad & \le \quad \int \rho_i^{\mathrm{c}_2}(x) U_{\rho_i^{\mathrm{e}} + \rho_i^{\mathrm{c}_1}}(x) \mathrm{d}x \le ||\rho_i||_{4/3} ||U_{\rho_i^{\mathrm{e}} + \rho_i^{\mathrm{c}_1}}||_{L^4(\{|x| \ge N_0^4\})} \\ & \le \quad C N_0^{1/2} \left(\int_{N_0^4}^{\infty} |x|^{-3} \mathrm{d}|x| \right)^{1/4} \le C N_0^{-3/2}, \end{aligned}$$

where we used the estimate

$$\begin{aligned} |U_{\rho_i^{\mathrm{e}} + \rho_i^{\mathrm{c}_1}}(x)| \quad & \le \quad \int_{|y| \le N_0} \frac{\rho_i(y)}{|x-y|} \mathrm{d}y \le \frac{1}{|x| - N_0} ||\rho_i||_{L^1(\{|x| \le N_0\})} \\ & \le \quad \frac{C ||\rho_i||_{4/3} N_0^{1/2}}{|x|}, \qquad |x| > 2N_0. \end{aligned}$$

Without loss of generality we may by (4.18) assume that

$$0 < \varepsilon_0/2 \leq \int_{|x|\geq N_0^4} \int f_i^{3/2} \mathrm{d}v\mathrm{d}x \leq M/2.$$

Now by the scaling equation (4.11) and the fact that $h_M < 0$ we have

$$\begin{aligned} \mathcal{H}(f_i) &\geq \left[\left(\frac{||f_i^{\mathrm{e}}||_{3/2}^{3/2}}{M} \right)^2 + \left(\frac{||f_i^{\mathrm{s}}||_{3/2}^{3/2}}{M} \right)^2 + \left(\frac{||f_i^{\mathrm{c}_1}||_{3/2}^{3/2}}{M} \right)^2 + \left(\frac{||f_i^{\mathrm{c}_2}||_{3/2}^{3/2}}{M} \right)^2 \right] h_M \\ &\quad - C\varepsilon^{1/2} - CN_0^{-1/2} \\ &\geq \left[\left(1 - \frac{||f_i^{\mathrm{c}_2}||_{3/2}^{3/2}}{M} \right)^2 + \left(\frac{||f_i^{\mathrm{c}_2}||_{3/2}^{3/2}}{M} \right)^2 \right] h_M - C\varepsilon^{1/2} - CN_0^{-1/2}, \end{aligned}$$

which is a contradiction with (f_i) being a minimizing sequence. Notice that we may choose ε sufficiently small and N_0 sufficiently large. Now having established the claim about the concentration we can finally prove the convergence of the potential energy term:

$$\begin{aligned} ||f_i - f_0||_{\mathrm{pot}} &\leq ||\mathbf{1}_{|x|\leq 1/N} f_i - \mathbf{1}_{|x|\leq 1/N} f_0||_{\mathrm{pot}} \\ &\quad + ||\mathbf{1}_{1/N\leq|x|\leq N} f_i - \mathbf{1}_{1/N\leq|x|\leq N} f_0||_{\mathrm{pot}} \\ &\quad + ||\mathbf{1}_{|x|\geq N} f_i - \mathbf{1}_{|x|\geq N} f_0||_{\mathrm{pot}}. \end{aligned}$$

Both the first and third term are uniformly small if N is large due to (4.17). For now fixed N the second term goes to zero by Lemma 4.7. Hence

$$E_{\mathrm{pot}}(f_i - f_0) \to 0.$$

Therefore we have, letting $i \to \infty$ and observing (4.15)

$$\mathcal{H}(f_0) \leq h_M.$$

In order for $f_0 \in \mathcal{F}_M$ we must have $\iint f_0^{3/2} = M$. Suppose on the contrary that $\iint f_0^{3/2} = \bar{M} < M$. Using the scaling $\bar{f}_0(x,v) := f_0(x, (\bar{M}/M)^{1/2} v)$ we have from the scaling equality (4.12):

$$\mathcal{H}(\bar{f}_0) = \left(\frac{\bar{M}}{M} \right)^{-2} \mathcal{H}(f_0) = \left(\frac{\bar{M}}{M} \right)^{-2} h_M < h_M,$$

and $\bar{f}_0 \in \mathcal{F}_M$, which is contradiction and $f_0 \in \mathcal{F}_M$. □

In Lemma 4.7 and 4.8 we needed the symmetry assumption in order to prove the convergence of the potential energy terms along the minimizing sequence. This would of course limit our stability result to stability against axially symmetric perturbations only.

To overcome this shortcoming we have to generalize the compactness properties of the potential energy for general minimizing sequences. We use the same trick as in [16] and define the for the cut-off parameter $N > 1$,

$$h_{|N}(x,v) := \begin{cases} h(x,v) & \text{if } 1/N \leq h(x,v) \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.9. *Let $(f_i) \subset \mathcal{F}_M$ be a minimizing sequence. Then there exists a sequence $(\lambda_i) \subset \mathbb{R}^+$ such that up to a subsequence the following holds:*

(a) *For any $\varepsilon > 0$ there exists some $N > 1$ such that for all sufficiently large $i \in \mathbb{N}$,*

$$||S_{\lambda_i} f_i - (S_{\lambda_i} f_i)_{|N}||_{3/2} < \varepsilon.$$

(b) *There exists a sequence $(a_i) \subset \mathbb{R}^2$ and $\varepsilon_0 > 0$, $R_0 > 0$ such that for all sufficiently large $i \in \mathbb{N}$,*

$$\int_{(a_i + B_{R_0}) \times \mathbb{R}^2} (S_{\lambda_i} f_i)^{3/2} \mathrm{d}x \mathrm{d}v \geq \varepsilon_0.$$

(c) *Let $g_i := T_{a_i} S_{\lambda_i} f_i$, $g_i \rightharpoonup g_0$ weakly in $L^{3/2}(\mathbb{R}^4)$ and $\rho_i := \rho_{g_i} \rightharpoonup \rho_0$ weakly in $L^{4/3}(\mathbb{R}^2)$. Then for any $R > 0$*

$$E_{\text{pot}}(\mathbf{1}_{B_R} \rho_i - \mathbf{1}_{B_R} \rho_0) \to 0.$$

Proof. To prove (a) we use Theorem 4.8. We consider (f_i^*), the sequence of axially symmetric rearrangements with respect to x of (f_i), which is again minimizing in $\mathcal{F}_M$. By Theorem 4.8 there exists a symmetric minimizer g such that $S_{\lambda_i} f_i^* \rightharpoonup g$ weakly in $L^{3/2}(\mathbb{R}^4)$. Since $||S_{\lambda_i} f_i^*||_{3/2} = ||f_i^*||_{3/2} = M = ||g||_{3/2}$ it follows that $S_{\lambda_i} f_i^* \to g$ strongly in $L^{3/2}(\mathbb{R}^4)$. Now since $S_{\lambda_i} f_i^*$ is axially symmetric and decreasing in $|x|$ and equi-measurable with $S_{\lambda_i} f_i$ we have $S_{\lambda_i} f_i^* = (S_{\lambda_i} f_i)^*$. For notational purposes we define $g_i := S_{\lambda_i} f_i$ in the proofs of (a) and (b).

Now for $\varepsilon > 0$ we chose $N > 1$ such that $||g||_{L^{3/2}(A_N)} < \varepsilon/2$ where $A_N := \{g \leq 1/N \vee g \geq N\}$. Let $A_{i,N} := \{g_i^* \leq 1/N \vee g_i^* \geq N\}$. Then for i sufficiently large,

$$\begin{aligned} ||g_i^* - g_i^*{}_{|N}||_{3/2} &= ||g_i^*||_{L^{3/2}(A_{i,N})} \leq ||g_i^* - g||_{L^{3/2}(A_{i,N})} + ||g||_{L^{3/2}(A_{i,N})} \\ &\leq \varepsilon/2 + ||g||_{L^{3/2}(A_{i,N})}. \end{aligned}$$

Up to a subsequence, $g_i^* \to g$ pointwise almost everywhere, so $\limsup_{i\to\infty} \mathbf{1}_{A_{i,N}} \leq \mathbf{1}_{A_N}$ a.e. and by Fatou's lemma

$$\limsup_{i\to\infty} ||g||_{L^{3/2}(A_{i,N})} \leq ||g||_{L^{3/2}(A_N)} < \varepsilon/2.$$

So up to a subsequence we have for sufficiently large i

$$||g_i^* - g_i^*{}_{|N}||_{3/2} < \varepsilon.$$

For any function $h \geq 0$ and $p \geq 1$ we have the identity

$$\begin{aligned}
\int (h - h_{|N})^p &= p \int_0^\infty s^{p-1} \mu\{h \mathbf{1}_{\{h<1/N \vee h>N\}} > s\} \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} \left[\mu\{s < h < 1/N\} + \mu\{h > \max\{s, N\}\}\right] \mathrm{d}s \\
&= p \int_0^\infty s^{p-1} \left[\mu\{s < h^* < 1/N\} + \mu\{h^* > \max\{s, N\}\}\right] \mathrm{d}s \\
&= \int (h^* - {h^*}_{|N})^p.
\end{aligned}$$

Now if we take $h = g_i = S_{\lambda_i} f_i$ then with the help of $S_{\lambda_i} f_i^* = (S_{\lambda_i} f_i)^*$ we have (a).

To prove (b) we split for $N > 1$, $g_i = g_{i|N} + (g_i - g_{i|N})$ and proceed as in Lemma 4 in [16]. Firstly we define for $R > 1$

$$K_R(x) := \begin{cases} R & , \ |x| < 1/R, \\ 1/|x| & , \ 1/R \leq |x| \leq R, \\ 0 & , \ |x| > R, \end{cases}$$

and

$$F_R(x) := \frac{1}{|x|} \mathbf{1}_{\{|x|>R\}}(x), \ G_R(x) := \left(\frac{1}{|x|} - R\right) \mathbf{1}_{\{|x|<1/R\}}(x).$$

It is easy to check that the latter split gives

$$\frac{1}{|x|} = K_R(x) + F_R(x) + G_R(x). \tag{4.21}$$

Now according to (4.21) we split the potential energy as

$$\iint \frac{\rho_i(x)\rho_i(y)}{|x-y|} \mathrm{d}x\mathrm{d}y = I_1 + I_2 + I_3,$$

where $\rho_i := \rho_{g_{i|N}}$. We estimate the parts as follows

$$\begin{aligned}
|I_1| &\leq R \iint_{|x-y|<R} \rho_i(x)\rho_i(y) \mathrm{d}x\mathrm{d}y \leq R C_N \sup_{y \in \mathbb{R}^2} \int_{y+B_R} \rho_i(x) \mathrm{d}x \\
&\leq C_N R^{3/2} \sup_{y \in \mathbb{R}^2} \left[\int_{y+B_R} \rho_i^{4/3} \mathrm{d}x\right]^{3/4} \\
&\leq C_N R^{3/2} \sup_{y \in \mathbb{R}^2} \left[\int_{y+B_R} \int g_{i|N}^{3/2} \mathrm{d}v\mathrm{d}x\right]^{1/2}, \\
|I_2| &\leq \frac{1}{R} \iint \rho_i(x)\rho_i(y) \mathrm{d}x\mathrm{d}y \leq C_N R^{-1}, \\
|I_3| &\leq ||\rho_i||_{3/2} ||\rho_i * G_R||_3 \leq C ||G_R||_{3/2} \leq C_N R^{-1/3}.
\end{aligned}$$

In the last estimate we used the boundedness of ρ_i in $L^{3/2}(\mathbb{R}^2)$ which comes from the boundedness of $g_{i|N}$ in $L^{3/2}\cap L^\infty(\mathbb{R}^4)$. Now since g_i is a minimizing sequence we have for any $R>1$ and i sufficiently large

$$\begin{aligned}
h_M/2 &> \mathcal{H}(g_i) \geq E_{\mathrm{pot}}(g_i) = E_{\mathrm{pot}}(g_{i|N} + (g_i - g_{i|N})) \\
&= E_{\mathrm{pot}}(g_{i|N}) + E_{\mathrm{pot}}(g_i - g_{i|N}) - 2\iint \frac{\rho_{g_{i|N}}(x)\rho_{g_i - g_{i|N}}(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \\
&\geq E_{\mathrm{pot}}(g_{i|N}) + E_{\mathrm{pot}}(g_i - g_{i|N}) - C||g_i - g_{i|N}||_{3/2} \\
&\geq -(|I_1| + |I_2| + |I_3|) + E_{\mathrm{pot}}(g_i - g_{i|N}) - C||g_i - g_{i|N}||_{3/2}.
\end{aligned}$$

Now we have

$$\begin{aligned}
\liminf_{i\to\infty}\left[\sup_{y\in\mathbb{R}^2}\int_{y+B_R}\int g_i^{3/2}\mathrm{d}v\mathrm{d}x\right]^{1/2} &\geq \liminf_{i\to\infty}\left[\sup_{y\in\mathbb{R}^2}\int_{y+B_R}\int g_{i|N}^{3/2}\mathrm{d}v\mathrm{d}x\right]^{1/2} \\
&\geq C_N \liminf_{i\to\infty} |I_1| R^{-3/2} \\
&\geq C_N R^{-3/2}\Big[-h_M/2 - R^{-1} - R^{-1/3} \\
&\quad + E_{\mathrm{pot}}(g_i - g_{i|N}) - C||g_i - g_{i|N}||_{3/2}\Big].
\end{aligned}$$

Now from part (a) and Lemma 4.2 we can choose $N>0$ such that $E_{\mathrm{pot}}(g_i - g_{i|N}) - C||g_i - g_{i|N}||_{3/2} > h_M/4$. Then we choose $R>0$ large enough so that the term in the brackets is positive, and we have (b). As to (c), it is enough to prove that

$$||g_i - g_0||_{\mathrm{pot}(B_R\times B_R)} \to 0, \quad i\to\infty,$$

where we use the notation

$$\langle \xi, \psi\rangle_{\mathrm{pot}(M)} := \frac{1}{2}\iint_M \frac{\rho_\xi(x)\rho_\psi(y)}{|x-y|}\mathrm{d}x\mathrm{d}y, \qquad ||\xi||_{\mathrm{pot}(M)} := \langle\xi,\xi\rangle_{\mathrm{pot}(M)}^{1/2},$$

where $B_R = \{x\in\mathbb{R}^2; |x|<R\}$. We split for any $\delta>0$ the integration domain into two parts:

$$\begin{aligned}
M_1 &:= \{(x,y)\in\mathbb{R}^4; |x|<R, |y|<R\}\cap\{|x-y|<\delta\}, \\
M_2 &:= \{(x,y)\in\mathbb{R}^4; |x|<R, |y|<R\}\cap\{|x-y|\geq\delta\}.
\end{aligned}$$

We estimate the first part as follows:

$$
\begin{aligned}
||g_i - g_0||_{\mathrm{pot}(M_1)} &\leq ||g_i - g_{i|N}||_{\mathrm{pot}(M_1)} + ||g_{i|N} - g_{0|N}||_{\mathrm{pot}(M_1)} \\
&\quad + ||g_{0|N} - g_0||_{\mathrm{pot}(M_1)} \\
&\leq ||g_{i|N} - g_{0|N}||_{\mathrm{pot}(M_1)} + C||g_i - g_{i|N}||_{3/2} + C||g_{0|N} - g_0||_{3/2} \\
&\leq ||\rho_{g_{i|N} - g_{0|N}}||_{3/2} ||\mathbf{1}_{B_\delta}| \cdot |^{-1}||_{3/2}^{1/2} + C||g_i - g_{i|N}||_{3/2} \\
&\quad + C||g_{0|N} - g_0||_{3/2} \\
&\leq C_N \left(\int_0^\delta r^{-1/2} \mathrm{d}r \right)^{1/3} + C||g_i - g_{i|N}||_{3/2} \qquad (4.22) \\
&\quad + C||g_{0|N} - g_0||_{3/2} \\
&= C_N \delta^{1/6} + C||g_i - g_{i|N}||_{3/2} + C||g_{0|N} - g_0||_{3/2}. \qquad (4.23)
\end{aligned}
$$

Next we estimate the second part:

$$
\begin{aligned}
\frac{1}{2}||g_i - g_0||_{\mathrm{pot}(M_2)} &= \left| \iint_{M_2} \frac{\rho_{g_i - g_0}(x)\rho_{g_i - g_0}(y)}{|x-y|} \mathrm{d}x\mathrm{d}y \right| \\
&= \left| \int \rho_{g_i - g_0}(x) h_i(x) \mathrm{d}x \right| \leq ||\rho_{g_i - g_0}||_{4/3} ||h_i||_4 \leq C||h_i||_4,
\end{aligned}
$$

where

$$
h_i(x) := \mathbf{1}_{B_R(x)} \int_{|x-y| \geq \delta} \mathbf{1}_{B_R(y)} \frac{1}{|x-y|} \rho_{g_i - g_0}(y) \mathrm{d}y \to 0.
$$

The reason for the above convergence is weak convergence of $\rho_{g_i - g_0}$ and the fact that the test function against which is $\rho_{g_i - g_0}$ tested lies in $L^4(\mathbb{R}^2)$. Now since $|h_i| \leq \mathbf{1}_{B_R} C_M / \delta$, converges $h_i \to 0$ in $L^4(\mathbb{R}^2)$ by Lebesgue's dominated convergence theorem. The majorising function for h_i of course does not depend on i, since from (a) we can find such N that the last two terms in (4.23) remain small for all i sufficiently large and then we can fix δ to control C_N. Now with δ being fixed we can pass through the dominated convergence argument. □

The existence theorem for the minimizer in the general class $\mathcal{F}_M$ is the following:

Theorem 4.10. *Let $(f_i) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}$. Then there exists a minimizer $f_0 \in \mathcal{F}_M$, a subsequence (also denoted by f_i), a sequence of translations $T_{a_i} f_i(x, v) = f_i(x + a_i, v)$ with $(a_i) \subset \mathbb{R}^2$ and a sequence of scalings S_{λ_i} with $\lambda_i > 0$ such that*

$$
\begin{gathered}
\mathcal{H}(f_0) = \inf_{\mathcal{F}_M} \mathcal{H} = h_M, \\
T_{a_i} S_{\lambda_i} f_i \rightharpoonup f_0 \text{ weakly in } L^{3/2}(\mathbb{R}^4), \\
E_{\mathrm{pot}}(T_{a_i} S_{\lambda_i} f_i - f_0) \to 0.
\end{gathered}
$$

Proof. We choose $(\lambda_i), (a_i), \varepsilon_0 > 0, R_0 > 0$ according to Lemma 4.9 and define $g_i := T_{a_i} S_{\lambda_i} f_i$, which is again a minimizing sequence in $\mathcal{F}_M$. This sequence is bounded in $L^{3/2}(\mathbb{R}^4)$ and by Lemma 4.2 the induced spatial densities ρ_i are bounded in $L^{4/3}(\mathbb{R}^2)$. Now we can pick subsequences (without changing the notation) such that

$$\begin{aligned} g_i &\rightharpoonup f_0 \quad \text{weakly in } L^{3/2}(\mathbb{R}^4), \\ \rho_i &\rightharpoonup \rho_0 := \rho_{f_0} \quad \text{weakly in } L^{4/3}(\mathbb{R}^2). \end{aligned}$$

By weak convergence and convexity of $||\cdot||_{3/2}^{3/2}$

$$E_{\mathrm{kin}}(f_0) \le \liminf_{i\to\infty} E_{\mathrm{kin}}(g_i) < \infty,$$

$$\iint f_0^{3/2} \mathrm{d}x\mathrm{d}v \le \liminf_{i\to\infty} \iint g_i^{3/2} \mathrm{d}x\mathrm{d}v = M.$$

We have to show the convergence of the potential energies. We use the method of splitting introduced in [16] and for $R > R_0$ we split g_i as follows:

$$\begin{aligned} g_i &= g_i \mathbf{1}_{B_{R_0}\times\mathbb{R}^2} + g_i \mathbf{1}_{B_{R_0,R}\times\mathbb{R}^2} + g_i \mathbf{1}_{B_{R,\infty}\times\mathbb{R}^2} \\ &:= g_i^{(1)} + g_i^{(2)} + g_i^{(3)}. \end{aligned} \tag{4.24}$$

From Lemma 4.9 we already know that $E_{\mathrm{pot}}(g_i^{(1)} + g_i^{(2)})$ converges to $E_{\mathrm{pot}}(g_0^{(1)} + g_0^{(2)})$ for any fixed R. So it is enough to show that for any $\varepsilon > 0$

$$\liminf_{i\to\infty} |E_{\mathrm{pot}}(g_i^{(3)})| < \varepsilon \tag{4.25}$$

for R large enough. By Lemma 4.2 it is sufficient to show that

$$\liminf_{i\to\infty} \iint \left(g_i^{(3)}\right)^{3/2} \mathrm{d}x\mathrm{d}v < \varepsilon. \tag{4.26}$$

To prove this we split the energy functional $\mathcal{H}$ according to (4.24):

$$\begin{aligned} \mathcal{H}(g_i) &= \mathcal{H}(g_i^{(1)}) + \mathcal{H}(g_i^{(2)}) + \mathcal{H}(g_i^{(3)}) \\ &\quad -2\langle g_i^{(2)}, g_i^{(1)} + g_i^{(3)}\rangle_{\mathrm{pot}} - 2\langle g_i^{(1)}, g_i^{(3)}\rangle_{\mathrm{pot}} \\ &=: \mathcal{H}(g_i^{(1)}) + \mathcal{H}(g_i^{(2)}) + \mathcal{H}(g_i^{(3)}) - I_i^1 - I_i^2. \end{aligned}$$

Now from the boundedness of $E_{\mathrm{pot}}(\rho_i^{(1)} + \rho_i^{(3)})$ we get the estimate

$$I_i^1 \le C||g_i^{(2)}||_{\mathrm{pot}}.$$

Since $\rho_i^{(2)}$ converges weakly in $L^{4/3}$ to $\rho_0^{(2)} := \rho_0 \mathbf{1}_{B_{R_0,R}}$ we have from Lemma 4.9 (c)

$$E_{\text{pot}}(\rho_i^{(2)} - \rho_0^{(2)}) \to 0, \quad i \to \infty. \tag{4.27}$$

The second mixed term I_i^2 we estimate for $R > 2R_0$ as follows:

$$I_i^2 \leq \int_{B_{R_0}} \rho_i(x)\mathrm{d}x \int_{B_{R,\infty}} |y|^{-1}\rho_i(y)\mathrm{d}y \leq C\left(\frac{R_0}{R}\right)^{1/2}.$$

Now from Lemma 4.11 we can write (with $M_i^{(\mathrm{m})} := ||g_i^{(\mathrm{m})}||_{3/2}^{3/2}, m = 1,2,3$)

$$\begin{aligned}
\mathcal{H}(g_i^{(1)}) + \mathcal{H}(g_i^{(2)}) + \mathcal{H}(g_i^{(3)}) &\geq h_{M_i^{(1)}} + h_{M_i^{(2)}} + h_{M_i^{(3)}} \\
&= \left[\left(\frac{M_i^{(1)}}{M}\right)^2 + \left(\frac{M_i^{(2)}}{M}\right)^2 + \left(\frac{M_i^{(3)}}{M}\right)^2\right] h_M \\
&\geq \left[\left(\frac{M_i^{(1)} + M_i^{(2)}}{M}\right)^2 + \left(\frac{M_i^{(3)}}{M}\right)^2\right] h_M \\
&= \left[1 - 2\frac{M_i^{(1)} + M_i^{(2)}}{M}\frac{M_i^{(3)}}{M}\right] h_M
\end{aligned}$$

and thus

$$\begin{aligned}
h_M - \mathcal{H}(g_i) - C_1 h_M M_i^{(1)} M_i^{(3)} &\leq I_i^1 + I_i^2 \\
&\leq C_2\left[||f_0^{(2)}||_{\text{pot}} + ||g_i^{(2)} - f_0^{(2)}||_{\text{pot}} + \left(\frac{R_0}{R}\right)^{1/2}\right].
\end{aligned}$$

Now suppose that (4.26) is false. Then there exists some $\varepsilon_1 > 0$ such that for every $R > 0$ and i large

$$\iint (g_i^{(3)})^{3/2}\mathrm{d}x\mathrm{d}v \geq \varepsilon_1.$$

Define

$$\varepsilon_2 := -C_1 h_M \varepsilon_0 \varepsilon_1 > 0,$$

where ε_0 is as in Lemma 4.9 (b) and then increase R_0 from that lemma such that $C_2||f_0^{(2)}||_{\text{pot}} \leq \varepsilon_2/4$. Next choose $R > 2R_0$ such that $C_2(R_0/R)^{1/2} \leq \varepsilon_2/4$. Then, for i large,

$$\begin{aligned}
h_M - \mathcal{H}(g_i) + \varepsilon_2 &\leq h_M - \mathcal{H}(g_i) - C_1 h_M M_i^{(1)} M_i^{(3)} \\
&\leq \frac{1}{2}\varepsilon_2 + C_2||g_i^{(2)} - f_0^{(2)}||_{\text{pot}}.
\end{aligned}$$

By (4.27) this contradicts the fact that (g_i) is minimizing. Thus (4.26) holds and (4.25) follows. Now it remains to show that $\iint f_0^{3/2}\mathrm{d}x\mathrm{d}v = M$. This follows exactly as in the proof of Theorem 4.8. $\square$

Now we can give the stability proof of the minimizer against general perturbations. But before that we need to define the appropriate distance on the phase space from the minimizer:

$$d(f, f_0) := \iint \left[\left(\frac{1}{2}|v|^2 + U_0\right)(f - f_0) - \frac{2}{3C_M}(f^{3/2} - f_0^{3/2})\right] \mathrm{d}v\mathrm{d}x,$$

where C_M is as in (4.4). It easy to check that $d(f, f_0) \geq 0$ for all $f \in \mathcal{F}_M$ and $d(f, f_0) = 0$ only if $f = f_0$. The energy difference of a state f from the minimizer f_0 is then

$$\begin{aligned} \mathcal{H}(f) - \mathcal{H}(f_0) &= \iint \left(\frac{1}{2}|v|^2 + U_0\right)(f - f_0)\mathrm{d}v\mathrm{d}x + E_{\mathrm{pot}}(f - f_0) \\ &= d(f, f_0) + E_{\mathrm{pot}}(f - f_0). \end{aligned} \tag{4.28}$$

Theorem 4.11. *Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}$ over $\mathcal{F}_M$ defined in (4.2). Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution $t \mapsto f(t)$ of the flat Vlasov-Poisson system with $f(0) \in C_c^2(\mathbb{R}^4) \cap \mathcal{F}_M$*

$$d(f(0), f_0) + |E_{\mathrm{pot}}(f(0) - f_0)| < \delta$$

implies that for every $t > 0$ as long as the classical solution exists there exist $\lambda > 0$ and $a \in \mathbb{R}^2$ so that

$$d(f(t), S_\lambda T_a f_0) + |E_{\mathrm{pot}}(f(t) - S_\lambda T_a f_0)| < \varepsilon.$$

Proof. Assume the assertion is false. Then we can find $\varepsilon_0 > 0$, solutions to the Vlasov-Poisson system f_i and times t_i such that

$$d(f_i(0), f_0) + |E_{\mathrm{pot}}(f_i(0) - f_0)| < 2^{-i}$$

and

$$d(f_i(t_i), S_\lambda T_a f_0) + |E_{\mathrm{pot}}(f_i(t_i) - S_\lambda T_a f_0)| \geq \varepsilon_0 \tag{4.29}$$

for every $\lambda > 0$ and $a \in \mathbb{R}^2$. Since the energy is preserved along classical solutions of the Vlasov-Poisson system, $(f_i(t_i))$ is also a minimizing sequence. Notice that by Theorem 4.1 the minimizer f_0 is unique up to translations and scalings. By the analysis above we have

$$E_{\mathrm{pot}}(f_i(t_i) - S_{\lambda_i} T_{a_i} f_0) \to 0.$$

Using (4.28)

$$d(f_i(t_i), S_{\lambda_i} T_{a_i} f_0) \to 0,$$

which is clearly a contradiction with (4.29). $\square$

5 Flat galaxies with dark matter halos

5.1 Motivation

Today, the existence of dark matter is considered as a well-respected theory in astrophysics. One of the evidences that speak for its existence is the analysis of the rotation curves in spiral galaxies. Around 1970 by that time widely accepted Keplerian extrapolation (with the fall-off in the rotation speed as $R^{-1/2}$ in the outer regions of the galaxy) was confronted with the improved sensitivity of observations showing that the profiles of the rotational velocities are rather flat with no signs of the proposed decay. One possible explanation was that there exists a massive halo of "undetectable matter" around the spiral galaxy which extends to larger radii then the optical disk, which would correct the profile from Keplerian to relatively flat (or even slightly increasing). This conclusion was first stated in [9]. For an introduction to the dark matter theory we refer to [3, Chapter 10] and references there.

We consider a model of this situation using the Vlasov-matter. According to our knowledge the first attempt to model galactic dark matter halos using the Vlasov-type matter was done in [28], where the Vlasov equation in the polytropic setup was used to investigate the structure of the halos. Here we follow a different approach. In [15, 16] a variational technique for the Vlasov-Poisson system was used to prove existence and non-linear stability of certain equilibria in galactic dynamics. In Chapter 3 and 4 a similar approach based on [21] was used for the flat Vlasov-Poisson system, which is in fact a singular case of the three dimensional case, where all the particles are concentrated in a plane. In this chapter we combine those two methods and prove the existence of stationary solution of a modified Vlasov-Poisson system, where both flat and normal distribution functions appear. The flat part will be used to model the normal "visible" matter and non-flat part for the dark matter. We would like to point out that we do not study the decoupled system with dark matter as an external potential as in [28]. Here, normal and dark matter evolve and interact with one another through the common potential which they create collectively.

We suppose that both normal and dark matter are of Vlasov-type and interact with each other through a gravitational potential given as a solution of the Poisson equation. The system of partial differential equations governing the evolution of such an ensemble can be (at least formally) written as:

$$\begin{aligned} \frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x U_{\mathrm{c}} \cdot \nabla_v f &= 0, && (5.1\mathrm{a}) \\ \frac{\partial \tilde{f}}{\partial t} + \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{f} - \nabla_{\tilde{x}} U_{\mathrm{c}}(\cdot, 0) \cdot \nabla_{\tilde{v}} \tilde{f} &= 0, && (5.1\mathrm{b}) \\ U_{\mathrm{c}}(x) := U_f(x) + \tilde{U}_{\tilde{f}}(x) &= - \int_{\mathbb{R}^3} \frac{\rho_f(y)}{|x-y|} \, \mathrm{d}y - \int_{\mathbb{R}^2} \frac{\tilde{\rho}_{\tilde{f}}(\tilde{y})}{|x - (\tilde{y}, 0)|} \, \mathrm{d}\tilde{y}, && (5.1\mathrm{c}) \\ \lim_{|x| \to \infty} U_{\mathrm{c}}(x) &= 0, && (5.1\mathrm{d}) \end{aligned}$$

where $f(t, x, v) : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^+$ (resp. $\tilde{f}(t, \tilde{x}, \tilde{v}) : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$) is the distribution function of dark (resp. normal) matter. Throughout this chapter we use the notation convention that all variables with (resp. without) tilde are used for flat (resp. non-flat) quantities. Also we use the simplified notation $\rho := \rho_f, \tilde{\rho} := \tilde{\rho}_{\tilde{f}}, \tilde{U} := \tilde{U}_{\tilde{f}}$ etc..

In Section 5.2 we present some integrability results for the flat potential. The Section 5.3 introduces the variational setup and presents a brief review of the existence results for the decoupled problems which play an important role throughout this chapter. In Section 5.4, where we prove some a-proiori estimate for the energy functional. Finally, the last section describes the properties and structure of the minimizer.

5.2 Integrability of the flat potential

In order to analyze the mixed term in (5.3), we need some information about integrability of the flat potential in $\mathbb{R}^3$. Some regularity result are presented in [21], but those are with respect to the plane $x_3 = 0$ and do not say anything about the quality of $\tilde{U}$ perpendicular to that plane.

Lemma 5.1. *Let $\tilde{\rho} \in L^{4/3}(\mathbb{R}^2)$. Then $\tilde{U} \in L^6(\mathbb{R}^3)$ with $\nabla \tilde{U} \in L^2(\mathbb{R}^3)$ and*

$$\begin{aligned} ||\tilde{U}||_{L^6(\mathbb{R}^3)} &\leq C ||\tilde{\rho}||_{L^{4/3}(\mathbb{R}^2)}, \\ ||\nabla \tilde{U}||_{L^2(\mathbb{R}^3)} &\leq C ||\tilde{\rho}||_{L^{4/3}(\mathbb{R}^2)}. \end{aligned}$$

Proof. We use the general form of Minkowski's inequality (see [19, p. 47]) and Young's inequality to obtain the first assertion:

$$\begin{aligned}
||\tilde{U}||_6^6 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}\frac{\tilde{\rho}(\tilde{y})}{|x-(\tilde{y},0)|}\,\mathrm{d}\tilde{y}\right)^6 \mathrm{d}x_3\,\mathrm{d}\tilde{x} \\
&\le \int\left[\int\left(\int\frac{\tilde{\rho}^6(\tilde{y})}{(|\tilde{x}-\tilde{y}|^2+x_3^2)^3}\,\mathrm{d}x_3\right)^{1/6}\mathrm{d}\tilde{y}\right]^6\mathrm{d}\tilde{x} \\
&= C\int\left[\int\frac{\tilde{\rho}(\tilde{y})}{|\tilde{x}-\tilde{y}|^{5/6}}\,\mathrm{d}\tilde{y}\right]^6\mathrm{d}\tilde{x} = ||\tilde{\rho}*|\cdot|^{-5/6}||_{L^6(\mathbb{R}^2)}^6 \\
&\le C||\tilde{\rho}||_{L^{4/3}(\mathbb{R}^2)}^6\,||\,|\cdot|^{-5/6}||_{L_{\mathrm{w}}^{12/5}(\mathbb{R}^2)}^6.
\end{aligned}$$

To prove the second estimate we simply repeat the above procedure for

$$\nabla\tilde{U}(x) = \int\frac{\tilde{\rho}(\tilde{y})(x-(\tilde{y},0))}{|x-(\tilde{y},0)|^3}\,\mathrm{d}\tilde{y}$$

and we get

$$\begin{aligned}
||\nabla\tilde{U}||_2^2 &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}\frac{\tilde{\rho}(\tilde{y})}{|x-(\tilde{y},0)|^2}\,\mathrm{d}\tilde{y}\right)^2 \mathrm{d}x_3\,\mathrm{d}\tilde{x} \\
&\le \int\left[\int\left(\int\frac{\tilde{\rho}^2(\tilde{y})}{(|\tilde{x}-\tilde{y}|^2+x_3^2)^2}\,\mathrm{d}x_3\right)^{1/2}\mathrm{d}\tilde{y}\right]^2\mathrm{d}\tilde{x} \\
&= C\int\left[\int\frac{\tilde{\rho}(\tilde{y})}{|\tilde{x}-\tilde{y}|^{3/2}}\,\mathrm{d}\tilde{y}\right]^2\mathrm{d}\tilde{x} = ||\tilde{\rho}*|\cdot|^{-3/2}||_{L^2(\mathbb{R}^2)}^2 \\
&\le C||\tilde{\rho}||_{L^{4/3}(\mathbb{R}^2)}^2\,||\,|\cdot|^{-3/2}||_{L_{\mathrm{w}}^{4/3}(\mathbb{R}^2)}^2.
\end{aligned}$$

In both calculations we used that $|\cdot|^{-\lambda} \in L_{\mathrm{w}}^{n/\lambda}(\mathbb{R}^n)$. □

5.3 Variational setup

Before we begin with the minimization procedure, we have to choose the appropriate functional to be minimized and a suitable minimization class of functions. In previous chapters we chose the total mechanical energy as our functional and as a class we chose all positive functions with some L^p and kinetic energy bounds. We will use the same strategy here. The total energy in this case is the sum of total energies of the individual components (flat galaxy and dark matter halo) plus the interaction potential energy, which comes from the fact, that galactic matter is under the influence of dark matter's gravity and vice versa. We can write

$$\mathcal{H}(f,\tilde{f}) = \mathcal{H}(f) + \tilde{\mathcal{H}}(\tilde{f}) + \frac{1}{2}\int\tilde{U}(x)\rho(x)\,\mathrm{d}x + \frac{1}{2}\int U(\tilde{x},0)\tilde{\rho}(\tilde{x})\,\mathrm{d}\tilde{x}.$$

Note that the last two integral make sense as long as we provide some integrability conditions (specified later) on f and $\tilde{f}$, the first one directly from Lemma 5.1 and the second one using the following estimate:

$$\begin{aligned} ||U(\cdot,0)||_{L^4(\mathbb{R}^2)} &\leq \sup_{\varphi\in L^{4/3}(\mathbb{R}^2)} \frac{\int |U(\tilde{x},0)\varphi(\tilde{x})|\,\mathrm{d}\tilde{x}}{||\varphi||_{4/3}} = \sup_{\varphi\in L^{4/3}(\mathbb{R}^2)} \frac{\int |\rho(x)\tilde{U}_\varphi(x)|\,\mathrm{d}x}{||\varphi||_{4/3}} \\ &\leq C||\rho||_{L^{6/5}(\mathbb{R}^3)}. \end{aligned}$$

In the calculation above we used the equality

$$\int |U(\tilde{x},0)\tilde{\rho}(\tilde{x})|\,\mathrm{d}\tilde{x} = \iint \frac{\rho(y)\tilde{\rho}(\tilde{x})}{|(\tilde{x},0)-y|}\,\mathrm{d}\tilde{x}\,\mathrm{d}y = \int |\tilde{U}(x)\rho(x)|\,\mathrm{d}x. \tag{5.2}$$

We will use the symmetry relation (5.2) to merge the mixed terms and minimize the total energy functional

$$\begin{aligned} \mathcal{H}(f,\tilde{f}) &:= \mathcal{H}(f)+\tilde{\mathcal{H}}(\tilde{f})+\int \tilde{U}\rho\,\mathrm{d}x \qquad (5.3) \\ &= E_{\mathrm{kin}}(f)+E_{\mathrm{pot}}(f)+\tilde{E}_{\mathrm{kin}}(\tilde{f})+\tilde{E}_{\mathrm{pot}}(\tilde{f})+\int \tilde{U}\rho\,\mathrm{d}x \end{aligned}$$

under the constraint

$$\begin{aligned} \mathcal{F}_{\mathbf{M}} := \Big\{(f,\tilde{f}) | f\in L^1_+(\mathbb{R}^6), \tilde{f}\in L^1_+(\mathbb{R}^4)\,, \quad & ||f||_1\leq M, ||f||_{1+1/k_1}\leq N, \\ & ||\tilde{f}||_1\leq \tilde{M}, ||\tilde{f}||_{1+1/k_2}\leq \tilde{N}, \\ & f(\tilde{x},x_3,\tilde{v},v_3)=f(\tilde{x},-x_3,\tilde{v},-v_3), \\ & E_{\mathrm{kin}}(f)+\tilde{E}_{\mathrm{kin}}(\tilde{f})<\infty\Big\}, \end{aligned}$$

where $\mathbf{M} := (M,N,\tilde{M},\tilde{N}) \in (\mathbb{R}^+)^4$ denotes the constraint vector, $0<k_1<7/2$, $0<k_2<2$ and

$$\begin{aligned} E_{\mathrm{kin}}(f) &:= \frac{1}{2}\iint |v|^2 f(x,v)\,\mathrm{d}x\,\mathrm{d}v, \\ \tilde{E}_{\mathrm{kin}}(\tilde{f}) &:= \frac{1}{2}\iint |\tilde{v}|^2 \tilde{f}(\tilde{x},\tilde{v})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{v}, \\ E_{\mathrm{pot}}(f) &:= -\frac{1}{2}\iint \frac{\rho_f(x)\rho_f(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y, \\ \tilde{E}_{\mathrm{pot}}(\tilde{f}) &:= -\frac{1}{2}\iint \frac{\tilde{\rho}_{\tilde{f}}(\tilde{x})\tilde{\rho}_{\tilde{f}}(\tilde{y})}{|\tilde{x}-\tilde{y}|}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} \end{aligned}$$

represent the total kinetic and potential energies. We denote by $\mathbf{M}^{\mathrm{3D}} := (M,N)$ the non-flat and by $\mathbf{M}^{\mathrm{FL}} := (\tilde{M},\tilde{N})$ the flat part of $\mathbf{M}$. The symmetry of f with respect to the x_1,x_2 and v_1,v_2 planes is necessary to maintain the galactic disk flat.

A lot of the properties derived in the next section depends on the existence and properties of so-called *decoupled* minimizers. This means minimizers of the variational problems, where one of the components is missing. Note, that this problem is not exactly the same as the one we discussed for example in Chapter 3. Here we have two separate L^p constrains and not a sum of those as it was in Chapter 3.

In the case of a trivial flat component, the existence of a minimizer was proved for example in [1]. We briefly repeat the proof for the other case of vanished non-flat component, since the method itself is analog. We want to find a minimizer of the energy functional $\tilde{\mathcal{H}}$ under the $\mathbf{M}^{\mathrm{FL}}$ constraint. With the help of the Riesz rearrangement inequality ([19, p. 87]) and the fact, that the kinetic energy as well as the constraints are invariant under the spherically symmetric rearrangement, we get that your problem is equivalent to the same problem, when the functions $\tilde{f}$ have the form

$$\tilde{f}(\tilde{x}, \tilde{v}) = \varphi(|\tilde{x}|, |\tilde{v}|),$$

where the function $\varphi : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is nonincreasing. Before we prove the actual existence, we need some preliminary estimates for $\tilde{f}$. From the monotonicity of $\tilde{f}$ and L^q and $\tilde{E}_{\mathrm{kin}}$ bounds for each $q \in [1, 1 + 1/k_2]$ we have

$$\begin{aligned}
\tilde{f}^q(\tilde{x}, \tilde{v})|\tilde{x}|^2|\tilde{v}|^2 &\leq C \int_0^{|\tilde{x}|} \int_0^{|\tilde{v}|} \varphi^q(r, s) rs \, \mathrm{d}s \, \mathrm{d}r \leq C \tilde{N}_q^q, \\
\tilde{f}(\tilde{x}, \tilde{v})|\tilde{x}|^2|\tilde{v}|^4 &\leq C \int_0^{|\tilde{x}|} \int_0^{|\tilde{v}|} \varphi(r, s) rs^3 \, \mathrm{d}s \, \mathrm{d}r \leq C \tilde{E}_{\mathrm{kin}}(\tilde{f}).
\end{aligned}$$

As a result, we immediately get the local bounds on the distribution function $\tilde{f}$:

$$\tilde{f}(\tilde{x}, \tilde{v}) \leq g(\tilde{x}, \tilde{v}) := \begin{cases} \frac{C}{|\tilde{x}|^{2/q}|\tilde{v}|^{2/q}}, & \text{for} \quad |\tilde{v}| \leq V(|\tilde{x}|), \\ \frac{C}{|\tilde{x}|^2|\tilde{v}|^4}, & \text{for} \quad |\tilde{v}| > V(|\tilde{x}|), \end{cases}$$

where $V(|\tilde{x}|) > 0$ is an arbitrary function. When we now want to calculate the spatial density ρ_g, we get

$$\begin{aligned}
\rho_g(\tilde{x}) &= \frac{C}{|\tilde{x}|^{2/q}} \int_0^{V(|\tilde{x}|)} \frac{|\tilde{v}|}{|\tilde{v}|^{2/q}} \, \mathrm{d}|\tilde{v}| + \frac{C}{|\tilde{x}|^2} \int_{V(|\tilde{x}|)}^{\infty} \frac{|\tilde{v}|}{|\tilde{v}|^4} \, \mathrm{d}|\tilde{v}| \\
&= \frac{C}{|\tilde{x}|^{2/q}} V^{2-2/q}(|\tilde{x}|) + \frac{C}{|\tilde{x}|^2 V^2(|\tilde{x}|)}.
\end{aligned}$$

For the choice

$$V(|\tilde{x}|) = V_q(|\tilde{x}|) := |\tilde{x}|^{(1-q)/(2q-1)}$$

we get

$$\rho_g(\tilde{x}) \leq |\tilde{x}|^{-2q/(2q-1)}$$

with the exponent $s := -2q/(2q-1)$ being such that

$$\begin{aligned} s &< -3/2, \quad \text{if} \quad 1 < q < 3/2, && \text{(5.4a)}\\ s &> -3/2, \quad \text{if} \quad q > 3/2. && \text{(5.4b)} \end{aligned}$$

Now we split our estimates for f by choosing $q = 1+1/k_2 > 3/2$ for $|\tilde{x}| \le 1$ and $q \in [1, 3/2]$ for $|\tilde{x}| > 1$:

$$\tilde{f}(\tilde{x}, \tilde{v}) \le g(\tilde{x}, \tilde{v}) := \begin{cases} \frac{C}{|\tilde{x}|^{2/(1+1/k_2)}|\tilde{v}|^{2/(1+1/k_2)}} & \text{for} \quad |\tilde{x}| \le 1 \quad \wedge \quad |\tilde{v}| \le V_{1+1/k_2}(|\tilde{x}|),\\ \frac{C}{|\tilde{x}|^2|\tilde{v}|^4} & \text{for} \quad |\tilde{x}| \le 1 \quad \wedge \quad |\tilde{v}| > V_{1+1/k_2}(|\tilde{x}|),\\ \frac{C}{|\tilde{x}|^{2/q}|\tilde{v}|^{2/q}} & \text{for} \quad |\tilde{x}| > 1 \quad \wedge \quad |\tilde{v}| \le V_q(|\tilde{x}|),\\ \frac{C}{|\tilde{x}|^2|\tilde{v}|^4} & \text{for} \quad |\tilde{x}| > 1 \quad \wedge \quad |\tilde{v}| > V_q(|\tilde{x}|). \end{cases}$$

Now from (5.4) we have

$$\rho_{\tilde{f}}(\tilde{x}) \le \rho_g(\tilde{x}) \le \begin{cases} Cr^{s_1} & \text{with } s_1 > -8/5 \quad \text{for } |\tilde{x}| \le 1,\\ Cr^{s_2} & \text{with } s_2 < -8/5 \quad \text{for } |\tilde{x}| > 1. \end{cases} \tag{5.5}$$

The crucial part is to prove the convergence of the potential energies along the minimizing sequence $\tilde{f}_n$ (see proof of Theorem 3.10). Since we have here the estimate $0 \le \tilde{f}_n \le g$, the finiteness of the potential energy for g would allow us to pass to limit using the dominated convergence theorem. But from the Hardy-Littlewood-Sobolev inequality we have

$$|\tilde{E}_{\text{pot}}(g)| \le C||\rho_g||^2_{4/3},$$

which is, thanks to the bound (5.5), finite.

5.4 Properties of $\mathcal{H}$

First we need to establish a lower bound on $\mathcal{H}$.

Lemma 5.2. *For all $(f, \tilde{f}) \in \mathcal{F}_{\mathbf{M}}$ the following holds:*

Let $n_1 = k_1 + 3/2, n_2 = k_2 + 1$. Then $\rho \in L^{1+1/n_1}(\mathbb{R}^3), \tilde{\rho} \in L^{1+1/n_2}(\mathbb{R}^2)$ with

$$\begin{aligned} ||\rho||_{1+1/n_1} &\le CN^{(k_1+1)/(n_1+1)} E_{\text{kin}}(f)^{3/(2k_1+5)},\\ ||\tilde{\rho}||_{1+1/n_2} &\le C\tilde{N}^{(k_2+1)/(n_2+1)} \tilde{E}_{\text{kin}}(\tilde{f})^{1/(k_2+2)},\\ -E_{\text{pot}}(f) &\le C||\rho||^2_{6/5} \le C_{\mathbf{M}} E_{\text{kin}}(f)^{1/2},\\ -\tilde{E}_{\text{pot}}(\tilde{f}) &\le C||\tilde{\rho}||^2_{4/3} \le C_{\mathbf{M}} \tilde{E}_{\text{kin}}(\tilde{f})^{1/2}. \end{aligned}$$

Proof. We use the classical optimization method (used for example in the proof of Lemma 4.2) to prove the first two inequalities. For fixed $R > 0$ we split the v-integral and using Holder's inequality and the definition of the kinetic energy we get

$$\begin{aligned} \rho(x) &= \int_{|v|\le R} f(x,v)\,\mathrm{d}v + \int_{|v|>R} f(x,v)\,\mathrm{d}v \\ &\le \left(\frac{4}{3}\pi R^3\right)^{1/(k_1+1)} \left(\int f^{1+1/k_1}(x,v)\,\mathrm{d}v\right)^{k_1/(k_1+1)} \\ &\quad + \frac{1}{R^2}\iint |v|^2 f(x,v)\,\mathrm{d}v\,\mathrm{d}x =: I(R). \end{aligned}$$

To obtain the optimal value of R we set the first derivative

$$\frac{\mathrm{d}I}{\mathrm{d}R} = 0.$$

After some algebra we get the first assertion. In the same manner we prove the flat version of this estimate. The last two inequalities follow using interpolation and the Hardy-Littlewood-Sobolev inequality. □

Lemma 5.3. *The functional $\mathcal{H}$ is bounded from below on $\mathcal{F}_{\mathbf{M}}$, i.e.*

$$0 > h_{\mathbf{M}} := \inf_{\mathcal{F}_{\mathbf{M}}} \mathcal{H} > -\infty.$$

Proof. We have for each $(f, \tilde{f}) \in \mathcal{F}_{\mathbf{M}}$

$$\begin{aligned} \left|\int \rho(x)\tilde{U}(x)\,\mathrm{d}x\right| &\le ||\rho||_{6/5}||\tilde{U}||_6 \le C||\rho||_{6/5}||\tilde{\rho}||_{4/3} \\ &= CE_{\mathrm{kin}}(f)^{1/4}\tilde{E}_{\mathrm{kin}}(\tilde{f})^{1/4} \le C_{\mathbf{M}}E_{\mathrm{kin}}(f)^{1/2} + C_{\mathbf{M}}\tilde{E}_{\mathrm{kin}}(\tilde{f})^{1/2}, \end{aligned}$$

$$\begin{aligned} \mathcal{H}(f,\tilde{f}) &\ge E_{\mathrm{kin}}(f) - C_{\mathbf{M}}E_{\mathrm{kin}}(f)^{1/2} + \tilde{E}_{\mathrm{kin}}(\tilde{f}) - C_{\mathbf{M}}\tilde{E}_{\mathrm{kin}}(\tilde{f})^{1/2} + \int U_{\tilde{f}}\rho\,\mathrm{d}x \\ &\ge E_{\mathrm{kin}}(f) - C_{\mathbf{M}}E_{\mathrm{kin}}(f)^{1/2} + \tilde{E}_{\mathrm{kin}}(\tilde{f}) - C_{\mathbf{M}}\tilde{E}_{\mathrm{kin}}(\tilde{f})^{1/2}. \end{aligned} \tag{5.6}$$

The negativeness of $h_{\mathbf{M}}$ is an easy consequence of

$$h_{\mathbf{M}} < \mathcal{H}(f_0^{3\mathrm{D}}, f_0^{\mathrm{FL}}) = \mathcal{H}(f_0^{3\mathrm{D}}) + \tilde{\mathcal{H}}(f_0^{\mathrm{FL}}) + \int \tilde{U}_0\rho_0\,\mathrm{d}x < 0.$$

□

Having established the lower bound on $\mathcal{H}$ we can choose a minimizing sequence

$$(f_n, \tilde{f}_n) \subset \mathcal{F}_{\mathbf{M}},\ \mathcal{H}(f_n, \tilde{f}_n) \to h_{\mathbf{M}}.$$

From (5.6) we get that both the kinetic and the potential energies are bounded along the minimizing sequence:

$$E_{\mathrm{kin}}(f_n) + \tilde{E}_{\mathrm{kin}}(\tilde{f}_n) + |E_{\mathrm{pot}}(f_n)| + |\tilde{E}_{\mathrm{pot}}(\tilde{f}_n)| \le C_{\mathbf{M}}. \tag{5.7}$$

The bound $C_{\mathbf{M}}$ could be find easily by putting $\mathcal{H}(f_n, \tilde{f}_n) \le 0$ (which is true for large n) in to the bound (5.6).

In order to analyze the mixed nonflat-flat terms, we need some information about how strong the mixed potential energy term is with respect to the potential energies of its individual components.

Lemma 5.4. *Let $\rho \in L_+^{6/5}(\mathbb{R}^3),\ \tilde{\rho} \in L_+^{4/3}(\mathbb{R}^2)$. Then*

$$\mathcal{I}(\rho, \tilde{\rho}) := \iint \frac{\rho(x)\tilde{\rho}(\tilde{y})}{|x - (\tilde{y}, 0)|}\, \mathrm{d}x\, \mathrm{d}\tilde{y} \le 2|E_{\mathrm{pot}}(\rho)|^{1/2} |\tilde{E}_{\mathrm{pot}}(\tilde{\rho})|^{1/2}.$$

Proof. First we show the assertion under the further assumption $\rho, \tilde{\rho} \in C_{\mathrm{c}}^{\infty}$. We have

$$\begin{aligned}
\mathcal{I}(\rho, \tilde{\rho}) &= \int \tilde{\rho}(\tilde{y})(\rho * |\cdot|^{-1})(\tilde{y}, 0)\, \mathrm{d}\tilde{y} = \int \tilde{\rho}(\tilde{y})(\rho * |\cdot|^{-1})(y)\, \mathrm{d}\delta_{y_3}\, \mathrm{d}\tilde{y} \\
&= \lim_{\varepsilon \to 0+} \int \tilde{\rho}(\tilde{y})(\rho * |\cdot|^{-1})(y)\, \mathrm{d}\delta_{y_3}^{\varepsilon}\, \mathrm{d}\tilde{y} = \lim_{\varepsilon \to 0+} \iint \frac{\rho(x)\tilde{\rho}(\tilde{y})\delta^{\varepsilon}(y_3)}{|x - y|}\, \mathrm{d}x\, \mathrm{d}y \\
&\le \left(\iint \frac{\rho(x)\rho(y)}{|x - y|}\, \mathrm{d}x \mathrm{d}y \right)^{1/2} \lim_{\varepsilon \to 0+} \left(\iint \frac{\tilde{\rho}(\tilde{x})\tilde{\rho}(\tilde{y})\delta^{\varepsilon}(x_3)\delta^{\varepsilon}(y_3)}{|x - y|}\, \mathrm{d}x \mathrm{d}y \right)^{1/2} \\
&\le \sqrt{2}|E_{\mathrm{pot}}(\rho)|^{1/2} \lim_{\varepsilon \to 0+} \left(\iint \frac{\tilde{\rho}(\tilde{x})\tilde{\rho}(\tilde{y})\delta^{\varepsilon}(x_3)\delta^{\varepsilon}(y_3)}{|\tilde{x} - \tilde{y}|}\, \mathrm{d}x\, \mathrm{d}y \right)^{1/2} \\
&= 2|E_{\mathrm{pot}}(\rho)|^{1/2} |\tilde{E}_{\mathrm{pot}}(\tilde{\rho})|^{1/2}.
\end{aligned}$$

Now when the smoothness of ρ and $\tilde{\rho}$ is not available, we use the standard regularization $\rho^{\varepsilon} := \rho * \delta^{\varepsilon}, \tilde{\rho}^{\varepsilon} := \tilde{\rho} * \tilde{\delta}^{\varepsilon}$ and we have:

$$\begin{aligned}
\mathcal{I}(\rho - \rho^{\varepsilon}, \tilde{\rho} - \tilde{\rho}^{\varepsilon}) &\le C||\rho - \rho^{\varepsilon}||_{6/5} ||\tilde{\rho} - \tilde{\rho}^{\varepsilon}||_{4/3} &\to 0, &\quad \varepsilon \to 0+, \\
\mathcal{I}(\rho, \tilde{\rho}) - \mathcal{I}(\rho^{\varepsilon}, \tilde{\rho}^{\varepsilon}) && \to 0, &\quad \varepsilon \to 0+
\end{aligned}$$

□

Now we present some classical compactness properties of the Poisson integral.

Theorem 5.5. *Let* $(\rho_n) \subset L^{1+1/n_1}(\mathbb{R}^3)$ *and* $(\tilde{\rho}_n) \subset L^{1+1/n_2}(\mathbb{R}^2)$ *be a bounded sequences with*

$$\begin{aligned} \rho_n &\rightharpoonup \rho_0 \quad \text{weakly} \quad \text{in } L^{1+1/n_1}(\mathbb{R}^3), \\ \tilde{\rho}_n &\rightharpoonup \tilde{\rho}_0 \quad \text{weakly} \quad \text{in } L^{1+1/n_2}(\mathbb{R}^2). \end{aligned}$$

Then for each $R > 0$ *we have*

$$\begin{aligned} E_{\text{pot}}(\mathbf{1}_{B_R}\rho_n - \mathbf{1}_{B_R}\rho_0) &\to 0, \\ \tilde{E}_{\text{pot}}(\mathbf{1}_{\tilde{B}_R}\tilde{\rho}_n - \mathbf{1}_{\tilde{B}_R}\tilde{\rho}_0) &\to 0, \\ \int (\mathbf{1}_{B_R}\rho_n - \mathbf{1}_{B_R}\rho_0)\tilde{U}_{\mathbf{1}_{\tilde{B}_R}\tilde{\rho}_n - \mathbf{1}_{\tilde{B}_R}\tilde{\rho}_0} \,\mathrm{d}x &\to 0. \end{aligned}$$

Proof. The convergence of non-flat potential energies was proved for example in [16] and for the flat case we can use Lemma 3.9. Finally, the convergence of the mixed term follows from Lemma 5.4. □

In the next lemma we explore concentration properties of a minimizing sequence.

Lemma 5.6. *Let* $(f_n, \tilde{f}_n) \subset \mathcal{F}_{\mathbf{M}}$ *be a minimizing sequence of* $\mathcal{H}$*. Then for all sufficiently large* $n \in \mathbb{N}$

$$\int \tilde{U}_n \rho_n \,\mathrm{d}x \quad < \quad -\frac{M\tilde{M}}{2(R_0^{\text{3D}} + R_0^{\text{FL}})}, \tag{5.8}$$

where R_0^{3D} *and* R_0^{FL} *are the radii of some chosen decoupled minimizers* f_0^{3D} *and* f_0^{FL} *subject to constraints* $\mathbf{M}^{\text{3D}}$ *and* $\mathbf{M}^{\text{FL}}$ *shifted so, that they are spherically (resp. axially) symmetric. Further we can find* $(\tilde{a}_n), (\tilde{b}_n) \subset \mathbb{R}^2$, $\varepsilon_0 > 0$ *and* $R_0 > 0$ *such that*

$$\int_{(\tilde{a}_n,0)+B_{R_0}} \int f_n \,\mathrm{d}v\,\mathrm{d}x \quad \geq \quad \varepsilon_0, \tag{5.9}$$

$$\int_{\tilde{b}_n+\tilde{B}_{R_0}} \int \tilde{f}_n \,\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{x} \quad \geq \quad \varepsilon_0 \tag{5.10}$$

for all sufficiently large $n \in \mathbb{N}$.

Proof. Assume that (5.8) is false. Then we have

$$\begin{aligned} \mathcal{H}(f_n, \tilde{f}_n) \quad &= \quad \mathcal{H}(f_n) + \tilde{\mathcal{H}}(\tilde{f}_n) + \int \tilde{U}_n \rho_n \,\mathrm{d}x \geq h_{\mathbf{M}^{\text{3D}}} + \tilde{h}_{\mathbf{M}^{\text{FL}}} - \frac{M\tilde{M}}{2(R_0^{\text{3D}} + R_0^{\text{FL}})} \\ &\geq \quad \mathcal{H}(f_0^{\text{3D}}) + \mathcal{H}(f_0^{\text{FL}}) + \frac{1}{2}\int \tilde{U}_0^{\text{FL}} \rho_0^{\text{3D}} \,\mathrm{d}x = \mathcal{H}(f_0^{\text{3D}}, f_0^{\text{FL}}) - \frac{1}{2}\int \tilde{U}_0^{\text{FL}} \rho_0^{\text{3D}} \,\mathrm{d}x \end{aligned}$$

which is clearly a contradiction. We used $h_{\mathbf{M}^{3D}}$ and $\tilde{h}_{\mathbf{M}^{FL}}$ as a symbols for the infimas of the energy functionals corresponding to the decoupled problems.

For $R > 1$ we split $|\cdot|^{-1}$ as

$$
\begin{aligned}
\frac{1}{|x|} &= K_R(x) + F_R(x) + G_R(x) \\
&:= \mathbf{1}_{\{1/R \le |x| \le R\}} \frac{1}{|x|} + \mathbf{1}_{\{|x| \ge R\}} \frac{1}{|x|} + \mathbf{1}_{\{|x| \le 1/R\}} \frac{1}{|x|}
\end{aligned}
$$

We use the latter splitting of the Poisson kernel to split the mixed potential energy as

$$
\left| \int \tilde{U}_n \rho_n \,\mathrm{d}x \right| = \iint \frac{\rho_n(x)\tilde{\rho}_n(\tilde{y})}{|x - (\tilde{y}, 0)|} \,\mathrm{d}x \,\mathrm{d}\tilde{y} = J_1 + J_2 + J_3.
$$

We have

$$
\begin{aligned}
|J_1| &\le R \iint_{|x-(\tilde{y},0)|<R} \rho_n(x)\tilde{\rho}_n(\tilde{y}) \,\mathrm{d}x \,\mathrm{d}\tilde{y} \le \tilde{M} R \sup_{\tilde{y}\in\mathbb{R}^2} \int_{(\tilde{y},0)+B_R} \int f_n(x,v) \,\mathrm{d}v \,\mathrm{d}x, \\
|J_1| &\le R \iint_{|\tilde{x}-\tilde{y}|<R} \rho_n(x)\tilde{\rho}_n(\tilde{y}) \,\mathrm{d}x \,\mathrm{d}\tilde{y} \le M R \sup_{\tilde{x}\in\mathbb{R}^2} \int_{\tilde{x}+\tilde{B}_R} \int \tilde{f}_n(\tilde{y},\tilde{v}) \,\mathrm{d}\tilde{v} \,\mathrm{d}\tilde{y}, \\
|J_2| &\le R^{-1} \iint \rho_n(x)\tilde{\rho}_n(\tilde{y}) \,\mathrm{d}x \,\mathrm{d}\tilde{y} \le M\tilde{M}R^{-1}, \\
|J_3| &\le ||\rho_n||_{1+1/n_1} \left|\left| \int_{|\tilde{x}-\tilde{y}|<1/R} \frac{\tilde{\rho}_n(\tilde{y})}{|x-(\tilde{y},0)|} \,\mathrm{d}\tilde{y} \right|\right|_{n_1+1} \\
&\le C||\rho_n||_{1+1/n_1} ||\tilde{\rho}_n * (\tilde{G}_R)^{n_1/(n_1+1)}||_{n_1+1} \\
&\le C||\rho_n||_{1+1/n_1} ||\tilde{\rho}_n||_{1+1/n_2} ||(\tilde{G}_R)^{n_1/(n_1+1)}||_{\gamma} \\
&\le C||\rho_n||_{1+1/n_1} ||\tilde{\rho}_n||_{1+1/n_2} R^{-\sigma} \le C_{\mathbf{M}} R^{-\sigma}
\end{aligned}
$$

with γ and σ defined as

$$
1 < \gamma := \left(\frac{1}{n_1+1} + \frac{1}{n_2+1} \right)^{-1}, \tag{5.11}
$$

$$
0 < \sigma := \frac{2}{\gamma} - \frac{n_1}{n_1+1}. \tag{5.12}
$$

We recall, that for the present choice of k_1 and k_2 we have $3/2 < n_1 < 5, 1 < n_2 < 3$. The last inequality in the J_3 estimate comes from the boundedness of $f_n, \tilde{f}_n$ using $\mathbf{M}$ and from the boundedness of kinetic energies (5.7). From (5.8) we have for n large

$$
-\frac{M\tilde{M}}{2(R_0^{3D} + R_0^{FL})} > \int \tilde{U}_n \rho_n \,\mathrm{d}x = -|J_1| - |J_2| - |J_3|.
$$

Hence

$$\begin{aligned} |J_1| &\geq \frac{M\tilde{M}}{2(R_0^{3\mathrm{D}}+R_0^{\mathrm{FL}})} - \frac{M\tilde{M}}{R} - C_{\mathbf{M}}R^{-\sigma}, \\ \sup_{\tilde{y}\in\mathbb{R}^2}\int_{(\tilde{y},0)+B_R}\int f_n \,\mathrm{d}v\,\mathrm{d}x &\geq R^{-1}\left(\frac{M}{2(R_0^{3\mathrm{D}}+R_0^{\mathrm{FL}})} - \frac{M}{R} - \frac{C_{\mathbf{M}}}{\tilde{M}}R^{-\sigma}\right). \end{aligned} \tag{5.13}$$

For R sufficiently large the right hand side is positive, which proves (5.9). When we repeat the same procedure, but using the second estimate of J_1, we get

$$\sup_{\tilde{x}\in\mathbb{R}^2}\int_{\tilde{x}+\tilde{B}_R}\int \tilde{f}_n \,\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{y} \geq R^{-1}\left(\frac{\tilde{M}}{2(R_0^{3\mathrm{D}}+R_0^{\mathrm{FL}})} - \frac{\tilde{M}}{R} - \frac{C_{\mathbf{M}}}{M}R^{-\sigma}\right), \tag{5.14}$$

which in turn gives (5.10). $\square$

The expressions (5.13) and (5.14) allow us to formulate the following Corollary:

Corollary 5.7. *Let the constraint vector* $\mathbf{M}$ *satisfies* $0 < M_0 \leq M \leq M_1, 0 < \tilde{M}_0 \leq \tilde{M} \leq \tilde{M}_1, 0 < N_0 \leq N \leq N_1, 0 < \tilde{N}_0 \leq \tilde{N} \leq \tilde{N}_1$. *Then the constants* ε_0 *and* R_0 *in (5.9) and (5.10) can be chosen independently of* $\mathbf{M}$ *and* $(f_n, \tilde{f}_n)$*, depending only on the bounds* $\mathbf{M}_0$ *and* $\mathbf{M}_1$.

Proof. From (5.13) and (5.14) we have for n sufficiently large

$$\begin{aligned} \sup_{\tilde{y}\in\mathbb{R}^2}\int_{(\tilde{y},0)+B_R}\int f_n \,\mathrm{d}v\,\mathrm{d}x &\geq R^{-1}\left(\frac{M_0}{2(R_{\max}^{3\mathrm{D}}+R_{\max}^{\mathrm{FL}})} - \frac{M_1}{R} - \frac{C_{\mathbf{M}_1}}{M_0}R^{-\sigma}\right), \\ \sup_{\tilde{x}\in\mathbb{R}^2}\int_{\tilde{x}+\tilde{B}_R}\int \tilde{f}_n \,\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{y} &\geq R^{-1}\left(\frac{\tilde{M}_0}{2(R_{\max}^{3\mathrm{D}}+R_{\max}^{\mathrm{FL}})} - \frac{\tilde{M}_1}{R} - \frac{C_{\mathbf{M}_1}}{\tilde{M}_0}R^{-\sigma}\right), \end{aligned}$$

where $R_{\max}^{3\mathrm{D}}$ and $R_{\max}^{\mathrm{FL}}$ are maximal radii of the decoupled minimizers with all possible constraints $\mathbf{M}$ satisfying the bound above. Note that these maximum radii are as well functions of $\mathbf{M}_0, \mathbf{M}_1$ and are bounded and $R_{\max}^{3\mathrm{D}}$ is away from zero. The boundedness can be proved by the classical scaling and splitting argument with (see [21, Lemma 4,5,6]) $\square$

Definition 5.8. *We say, that the constraint vector* $\mathbf{M} \in (\mathbb{R}_0^+)^4$ *has at least one nontrivial component, when*

$$(M > 0 \wedge N > 0) \vee (\tilde{M} > 0 \wedge \tilde{N} > 0).$$

Lemma 5.9. *Let* $\mathbf{M}_1, \mathbf{M}_2 \in (\mathbb{R}_0^+)^4$. *Then the following holds:*

$$h_{\mathbf{M_1}} + h_{\mathbf{M}_2} \geq h_{\mathbf{M}_1+\mathbf{M}_2}. \tag{5.15}$$

If both $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *have at least one nontrivial component, then there exists* ε *(depending only on* $\mathbf{M}_1$ *and* $\mathbf{M}_2$*) such that*

$$h_{\mathbf{M_1}} + h_{\mathbf{M}_2} \geq h_{\mathbf{M}_1+\mathbf{M}_2} + \varepsilon. \tag{5.16}$$

Proof. We construct two minimizing sequences with

$$\begin{aligned} \mathcal{H}(f_n^1, \tilde{f}_n^1) &\rightarrow h_{\mathbf{M}_1}, \\ \mathcal{H}(f_n^2, \tilde{f}_n^2) &\rightarrow h_{\mathbf{M}_2}. \end{aligned}$$

From the Minkowski's inequality we have $(f_n^1 + f_n^2, \tilde{f}_n^1 + \tilde{f}_n^2) \in \mathcal{F}_{\mathbf{M}_1+\mathbf{M}_2}$ which gives

$$\begin{aligned} h_{\mathbf{M}_1+\mathbf{M}_2} &\leq \mathcal{H}(f_n^1 + f_n^2, \tilde{f}_n^1 + \tilde{f}_n^2) = \mathcal{H}(f_n^1, \tilde{f}_n^1) + \mathcal{H}(f_n^2, \tilde{f}_n^2) - \iint \frac{\rho_n^1(x)\rho_n^2(y)}{|x-y|} \,\mathrm{d}x \,\mathrm{d}y \\ &\quad - \iint \frac{\tilde{\rho}_n^1(\tilde{x})\tilde{\rho}_n^2(\tilde{y})}{|\tilde{x}-\tilde{y}|} \,\mathrm{d}\tilde{x} \,\mathrm{d}\tilde{y} - \iint \frac{\rho_n^1(x)\tilde{\rho}_n^2(\tilde{y})}{|x-(\tilde{y},0)|} \,\mathrm{d}x \,\mathrm{d}\tilde{y} - \iint \frac{\rho_n^2(x)\tilde{\rho}_n^1(\tilde{y})}{|x-(\tilde{y},0)|} \,\mathrm{d}x \,\mathrm{d}\tilde{y} \\ &\leq \mathcal{H}(f_n^1, \tilde{f}_n^1) + \mathcal{H}(f_n^2, \tilde{f}_n^2) \rightarrow h_{\mathbf{M}^{\mathrm{3D}}} + h_{\mathbf{M}^{\mathrm{FL}}}. \end{aligned}$$

Now when $\mathbf{M}_1$ and $\mathbf{M}_2$ have each at least one nontrivial component, we can without loss on generality assume that sequences are already shifted so that the analogy of assertions (5.9) or (5.10) holds for $\varepsilon_0^1, \varepsilon_0^2, R_0^1$ and R_0^2 without spatial shifts for these nontrivial components. Then we pick from the above estimate that mixed term which contains the latter components. Here we demonstrate the case when both non-flat components are nontrivial:

$$\begin{aligned} h_{\mathbf{M}_1+\mathbf{M}_2} &\leq \mathcal{H}(f_n^1 + f_n^2, \tilde{f}_n^1 + \tilde{f}_n^2) \leq \mathcal{H}(f_n^1, \tilde{f}_n^1) + \mathcal{H}(f_n^2, \tilde{f}_n^2) - \iint \frac{\rho_n^1(x)\rho_n^2(y)}{|x-y|} \,\mathrm{d}x \,\mathrm{d}y \\ &\leq \mathcal{H}(f_n^1, \tilde{f}_n^1) + \mathcal{H}(f_n^2, \tilde{f}_n^2) - \iint\limits_{B_{R_0^1} \times B_{R_0^2}} \frac{\rho_n^1(x)\rho_n^2(y)}{|x-y|} \,\mathrm{d}x \,\mathrm{d}y \\ &\leq \mathcal{H}(f_n^1, \tilde{f}_n^1) + \mathcal{H}(f_n^2, \tilde{f}_n^2) - \frac{\varepsilon_0^1 \varepsilon_0^2}{R_0^1 + R_0^2} \rightarrow h_{\mathbf{M}_1} + h_{\mathbf{M}_2} - \frac{\varepsilon_0^1 \varepsilon_0^2}{R_0^1 + R_0^2}. \end{aligned}$$

□

Since ε from the previous lemma comes implicit from Lemma 5.6, the analogy of Corollary 5.7 holds for ε as well.

5.5 Properties of the minimizer

Say we have proved the existence of the minimizing element $(f_0, \tilde{f}_0)$ of the variational problem describen above. First we have to exclude the possibility that $f_0 \equiv 0$ or $\tilde{f}_0 \equiv 0$. The case with both parts vanishing we can rule out simply from $h_{\mathbf{M}} < 0$. The case when only one component vanishes cannot happen either, since we can substitute the trivial component by the minimizer from the decoupled problem and obtain a state with lower energy. The next lemma proves that at the constraints are always in some sense saturated by minimizer.

Lemma 5.10. *Let* $(f_0, \tilde{f}_0) \in \mathcal{F}_{\mathbf{M}}$ *be a minimizer of* $\mathcal{H}$ *over* $\mathcal{F}_{\mathbf{M}}$*. Then*

$$\begin{array}{rcl} ||f_0||_1 = M & \vee & ||\tilde{f}_0||_1 = \tilde{M}, \\ ||f_0||_{1+1/k_1} & = & N, \\ ||\tilde{f}_0||_{1+1/k_2} & = & \tilde{N}, \end{array}$$

Proof. Let us say for example $||f_0||_{1+1/k_1} < N$. We define for $a, b, c, d, e > 0$ a new, rescaled state $(f_0^*, \tilde{f}_0^*)$ as

$$\begin{array}{rcl} f_0^*(x, v) & := & a f_0(bx, cv), \\ \tilde{f}_0^*(\tilde{x}, \tilde{v}) & := & d\tilde{f}_0(b\tilde{x}, e\tilde{v}). \end{array}$$

For kinetic and potential energies holds:

$$\begin{array}{rcl} E_{\mathrm{kin}}(f_0^*) & = & ab^{-3}c^{-5}E_{\mathrm{kin}}(f_0), \\ E_{\mathrm{pot}}(f_0^*) & = & a^2b^{-5}c^{-6}E_{\mathrm{pot}}(f_0), \\ \tilde{E}_{\mathrm{kin}}(\tilde{f}_0^*) & = & db^{-2}e^{-4}\tilde{E}_{\mathrm{kin}}(\tilde{f}_0), \\ \tilde{E}_{\mathrm{pot}}(\tilde{f}_0^*) & = & d^2b^{-3}e^{-4}\tilde{E}_{\mathrm{pot}}(\tilde{f}_0), \\ \int \tilde{U}_0^* \rho_0^* \,\mathrm{d}x & = & adb^{-4}c^{-3}e^{-2} \int \tilde{U}_0 \rho_0 \,\mathrm{d}x. \end{array}$$

We know choose the parameters a, b, c, d, e so that

$$||f_0^*||_1 = ||f_0||_1, \tilde{\mathcal{H}}(\tilde{f}_0^*) = \tilde{\mathcal{H}}(\tilde{f}_0), \text{ and } \int \tilde{U}_0^* \rho_0^* \,\mathrm{d}x = \int \tilde{U}_0 \rho_0 \,\mathrm{d}x.$$

This is true for

$$a = \gamma^3, c = \gamma, b = d = e = 1, \quad \gamma > 0.$$

For this particular choice of a, b, c, d, e we have

$$\begin{array}{rcl} ||f_0^*||_1 & = & ||f_0||_1, \\ ||\tilde{f}_0^*||_1 & = & ||\tilde{f}_0||_1, \\ ||f_0^*||_{1+1/k_1} & = & \gamma^{3/(k_1+1)}||f_0||_{1+1/k_1}, \\ ||\tilde{f}_0^*||_{1+1/k_2} & = & ||\tilde{f}_0||_{1+1/k_2} \end{array}$$

and for the energy

$$\mathcal{H}(f_0^*, \tilde{f}_0^*) = \gamma^{-2}E_{\mathrm{kin}}(f_0) + E_{\mathrm{pot}}(f_0) + \tilde{\mathcal{H}}(\tilde{f}_0) + \int \tilde{U}_0 \rho_0 \,\mathrm{d}x.$$

Hence we can surely choose $\gamma > 1$ so that we get a state which still lies in $\mathcal{F}_{\mathbf{M}}$ and has lower energy. With the same argument we get saturation for the flat part as well, $||\tilde{f}_0||_{1+1/k_2} = \tilde{N}$. The last thing to prove is saturation of the minimizer in L^1 norm.

Assume that $||f_0||_1 < M \wedge ||\tilde{f}_0||_1 < \tilde{M}$. In the scaling introduced above we choose

$$a = \gamma^{-7}, b = d = \gamma^{-4}, c = e = \gamma, \quad \gamma > 0.$$

For this choice of scaling parameters we have

$$\begin{aligned} ||f_0^*||_1 &= \gamma^2 ||f_0||_1, \\ ||\tilde{f}_0^*||_1 &= \gamma^2 ||\tilde{f}_0||_1, \\ ||f_0^*||_{1+1/k_1} &= \gamma^{(2k_1-7)/(k_1+1)} ||f_0||_{1+1/k_1}, \\ ||\tilde{f}_0^*||_{1+1/k_2} &= \gamma^{(2k_2-4)/(k_2+1)} ||\tilde{f}_0||_{1+1/k_2}, \end{aligned}$$

and

$$\mathcal{H}(f_0^*, \tilde{f}_0^*) = \mathcal{H}(f_0, \tilde{f}_0).$$

We can now choose $\gamma > 1$ such that $(f_0^*, \tilde{f}_0^*) \in \mathcal{F}_{\mathbf{M}}$ and $\mathbf{M} - \mathbf{M^0}$ has at least one non-trivial component. Now we use the strict subadditivity (5.16) and get

$$h_{\mathbf{M}} < h_{\mathbf{M}^0} + h_{\mathbf{M}-\mathbf{M}^0},$$

which contradicts the fact $\mathcal{H}(f_0, \tilde{f}_0) = h_{\mathbf{M}}$. □

Finally, we prove that the minimizers of the energy functional are functions of a total mechanical energy. We use the classical Lagrange multiplier method presented for example in [15, 16, 27, 4].

Theorem 5.11. *Let $(f_0, \tilde{f}_0)$ be the minimizer obtained in the previous section with potentials $(U_0, \tilde{U}_0)$. Then*

$$\begin{aligned} f_0(x, v) &= \left(\frac{E_0 - E_c(x, v)}{\lambda}\right)_+^{k_1} \qquad \text{a.e.}, \\ \tilde{f}_0(\tilde{x}, \tilde{v}) &= \left(\frac{\tilde{E}_0 - E_c(\tilde{x}, 0, \tilde{v}, 0)}{\tilde{\lambda}}\right)_+^{k_2} \qquad \text{a.e.}, \end{aligned}$$

where $E_c(x, v) := \frac{1}{2}|v|^2 + U_0(x) + \tilde{U}_0(x)$. The Lagrange multipliers are defined as

$$\begin{aligned} E_0 &:= \frac{1}{||f_0||_1}\left(\frac{2k_1+5}{3}E_{\text{kin}}(f_0) + 2E_{\text{pot}}(f_0) + \int \rho_0 \tilde{U}_0 \,\mathrm{d}x\right) < 0, \\ \tilde{E}_0 &:= \frac{1}{||\tilde{f}_0||_1}\left((k_2+2)\tilde{E}_{\text{kin}}(\tilde{f}_0) + 2\tilde{E}_{\text{pot}}(\tilde{f}_0) + \int \rho_0 \tilde{U}_0 \,\mathrm{d}x\right) < 0, \\ \lambda &:= \frac{2(k_1+1)E_{\text{kin}}(f_0)}{3||f_0||_{1+1/k_1}^{1+1/k_1}} > 0, \\ \tilde{\lambda} &:= \frac{(k_2+1)\tilde{E}_{\text{kin}}(\tilde{f}_0)}{||\tilde{f}_0||_{1+1/k_2}^{1+1/k_2}} > 0. \end{aligned}$$

Proof. Let $(f_0, \tilde{f}_0)$ be a minimizer of $\mathcal{H}$ with corresponding potentials $(U_0, \tilde{U}_0)$. We first define two new functionals

$$\begin{aligned}\mathcal{G}(f) &:= \mathcal{H}(f, \tilde{f}_0),\\ \tilde{\mathcal{G}}(\tilde{f}) &:= \mathcal{H}(f_0, \tilde{f}).\end{aligned}$$

From the definition of $\mathcal{H}$ we get

$$\begin{aligned}\mathcal{G}(f) - \mathcal{G}(f_0) &= E_{\text{kin}}(f) - E_{\text{kin}}(f_0) + E_{\text{pot}}(f) - E_{\text{pot}}(f_0) + \iint (\rho_f - \rho_0)\tilde{U}_0 \,\mathrm{d}x, \quad (5.17)\\ \tilde{\mathcal{G}}(\tilde{f}) - \tilde{\mathcal{G}}(\tilde{f}_0) &= \tilde{E}_{\text{kin}}(\tilde{f}) - \tilde{E}_{\text{kin}}(\tilde{f}_0) + \tilde{E}_{\text{pot}}(\tilde{f}) - \tilde{E}_{\text{pot}}(\tilde{f}_0) + \iint (\tilde{\rho}_{\tilde{f}} - \tilde{\rho}_0)U_0(\cdot, 0) \,\mathrm{d}\tilde{x}.\end{aligned}$$

We now define for each fixed $\varepsilon > 0$ the set $S_\varepsilon := \{(x, y) \in \mathbb{R}^6 : \varepsilon \le f_0(x, v) \le \varepsilon^{-1}\}$. Now let $\eta \in L^\infty(\mathbb{R}^6)$ be a real-valued function with compat support such that $\eta \ge 0$ a.e. for $(x, v) \in \mathbb{R}^6 \backslash \operatorname{supp} f_0$ and $\operatorname{supp} \eta \subseteq (\mathbb{R}^6 \backslash \operatorname{supp} f_0) \cup S_\varepsilon$. For $t \in [0, T]$ and $T = (||\eta||_1 + ||\eta||_{1+1/k_1} + ||\eta||_\infty)^{-1}\varepsilon/2$ we define

$$f_t(x, v) := \alpha^3(t)||f_0||_1 \frac{f_0 + t\eta}{||f_0 + t\eta||_1}(x, \alpha(t)v),$$

where

$$\alpha(t) := \left(\frac{||f_0||_{1+1/k_1}}{||f_0||_1} \frac{||f_0 + t\eta||_1}{||f_0 + t\eta||_{1+1/k_1}}\right)^{(k_1+1)/3}.$$

Note that we have for $t \in [0, T]$

$$||f_t||_1 = ||f_0||_1, \qquad ||f_t||_{1+1/k_1} = ||f_0||_{1+1/k_1}$$

and that $f_0 + t\eta \ge 0$ a.e.. From the bounds (for ε small enough)

$$\frac{||f_0||_1}{2} \le ||f_0 + t\eta||_1 \le ||f_0||_1 + \frac{\varepsilon}{2}, \quad \frac{||f_0||_{1+1/k1}}{2} \le ||f_0 + t\eta||_{1+1/k_1} \le ||f_0||_{1+1/k_1} + \frac{\varepsilon}{2}$$

we can infer that α is a smooth function on $[0, T]$ and

$$\alpha'(t) = \frac{k_1 + 1}{3}\alpha(t)\left[\frac{||\eta||_1}{||f_0 + t\eta||_1} - \frac{\iint (f_0 + t\eta)^{1/k_1}\eta \,\mathrm{d}x \,\mathrm{d}v}{||f_0 + t\eta||_{1+1/k_1}^{1+1/k_1}}\right].$$

Moreover $\sup_{[0,T]} \alpha''(t)$ is bounded. Now we have from (5.17) for $t \in [0, T]$

$$\begin{aligned}\mathcal{G}(f_t) - \mathcal{G}(f_0) &= \left(\frac{||f_0||_1}{\alpha^2(t)||f_0 + t\eta||_1} - 1\right) E_{\text{kin}}(f_0) + \frac{||f_0||_1 t}{\alpha^2(t)||f_0 + t\eta||_1} E_{\text{kin}}(\eta) \quad (5.18)\\ &\quad + \left(\frac{||f_0||_1^2}{||f_0 + t\eta||_1^2} - 1\right) E_{\text{pot}}(f_0) + \frac{||f_0||_1^2 t}{||f_0 + t\eta||_1^2} \int \rho_\eta U_0 \,\mathrm{d}x\\ &\quad + \frac{||f_0||_1^2 t^2}{||f_0 + t\eta||_1^2} E_{\text{pot}}(\eta) + \left(\frac{||f_0||_1}{||f_0 + t\eta||_1} - 1\right) \int \rho_0 \tilde{U}_0 \,\mathrm{d}x\\ &\quad + \frac{||f_0||_1 t}{||f_0 + t\eta||_1} \int \rho_\eta \tilde{U}_0 \,\mathrm{d}x.\end{aligned}$$

By a Taylor expansion at $t = 0+$ we obtain

$$\begin{aligned}
\frac{||f_0||_1}{\alpha^2(t)||f_0 + t\eta||_1} - 1 &= -t\left[\frac{||\eta||_1}{||f_0||_1} + 2\frac{k_1+1}{3}\left(\frac{||\eta||_1}{||f_0||_1} - \frac{\iint f_0^{1/k_1}\eta \,\mathrm{d}x\,\mathrm{d}v}{||f_0||_{1+1/k_1}^{1+1/k_1}}\right)\right] + O(t^2),\\
\frac{||f_0||_1 t}{\alpha^2(t)||f_0 + t\eta||_1} &= t + O(t^2),\\
\frac{||f_0||_1^2}{||f_0 + t\eta||_1^2} - 1 &= -\frac{2||\eta||_1 t}{||f_0||_1} + O(t^2),\\
\frac{||f_0||_1^2 t}{||f_0 + t\eta||_1^2} &= t + O(t^2),\\
\frac{||f_0||_1}{||f_0 + t\eta||_1} - 1 &= -\frac{||\eta||_1 t}{||f_0||_1} + O(t^2),\\
\frac{||f_0||_1 t}{||f_0 + t\eta||_1} &= t + O(t^2),
\end{aligned}$$

where the notation $O(t^2)$ means that the rest terms are bounded by Ct^2 with C not depending on t. When we now substitute the above estimates into (5.18), we obtain

$$\mathcal{G}(f_t) - \mathcal{G}(f_0) = t\iint (E_\mathrm{c} - E_0 + cf_0^{1/k_1})\eta \,\mathrm{d}x\,\mathrm{d}v + O(t^2)$$

with E_0 and c given in Theorem 5.11. Recalling the free choice of η and letting $\varepsilon \to 0$ we get that $E - E_0 \geq 0$ on $\mathbb{R}^6\backslash\operatorname{supp} f_0$ and that

$$f_0 = \left(\frac{E_0 - E_\mathrm{c}}{\lambda}\right)^{k_1} \qquad \text{a.e. on } \operatorname{supp} f_0.$$

By repeating the same procedure for the functional $\tilde{\mathcal{G}}$ we get the assertion for $\tilde{f}_0$. □

Finally we would like to point out some remarks concerning this chapter:

1. Although we know that the minimizer must be a function of total mechanical energy, we have so far no existence result. The conjecture is that the existence of the energy minizer should be proved using the the concentration compactness argument (see for example [16]) as it was done in the decoupled problems. The key argument, convergence of the potential energies,is however in out combined setup more difficult. We need to prove that

$$\forall\varepsilon > 0\, \exists R > 0 : \liminf_{n\to\infty}\left(|E_\mathrm{pot}(f_n\mathbf{1}_{|x|>R})| + |\tilde{E}_\mathrm{pot}(\tilde{f}_n\mathbf{1}_{|\tilde{x}|>R})|\right) < \varepsilon,$$

which is beyond our reach for the moment.

2. Even if we had the existence result, we could not say that the minimizer is a stationary solution to (5.1). To do so, we need the potentials U_0 and $\tilde{U}_0$ to be sufficiently regular or we need another (perhaps weaker) sence of solutions which allows less regularity on the potentials. In this sence, we can call this minimizer a "conditional" steady-state of the combined Vlasov-Poisson system.

6 Further tasks and open problems

As in [5], we would like to close this thesis with a small set of (partially) open questions or problems, which are connected to the topic.

6.1 Numerical computation of flat stationary solutions

In the full three-dimensional case the numerical computation of stationary solutions of the Vlasov-Poisson system is very straightforward. The minimizing state of the energy Casimir functional and its spatial density must have in the isotropic case the form

$$\begin{aligned} f_0(x,v) &= (E_0 - E)_+^k, \\ \rho_0(x) &= C(E_0 - U_0)_+^{k+3/2}. \end{aligned}$$

The only thing to do now is to calculate the potential which solves the Poisson equation. In our case

$$\Delta U_0 = C(E_0 - U_0)_+^{k+3/2}. \tag{6.1}$$

Since the minimizer (and its potential) is spherically symmetric, we look only for radial solutions of (6.1). In spherical coordinates (6.1) becomes the following ordinary differential equation in $r = |x|$:

$$\frac{1}{r^2}(r^2 U_0')' = C(E_0 - U_0)_+^{k+3/2},$$

which can be numerically easily solved when a suitable initial condition on the potential U_0 is prescribed.

In the flat case the situation is slightly more complicated. We do not have the Poisson equation, instead of that we have for the potential U_0 the singular integral equation (2.2b) to solve. This is not very convenient from the numerical point of view since the support of U_0 is unbounded.

Instead of computing the potential U_0 we can try to compute the spatial density directly using the numerical minimization of the reduced energy-Casimir functional (3.1) under the total mass constraint. Since our problem is now again one-dimensional (we work with the axially symmetric functions), we used a modified steepest descent method to do the minimization.

Since the reduction is possible in the full three-dimensional case (see [24]), we can use this fact to test our method. However, the minimizer coming from the numerical minimization need not to be the same as the one obtaine by analytical methods. The

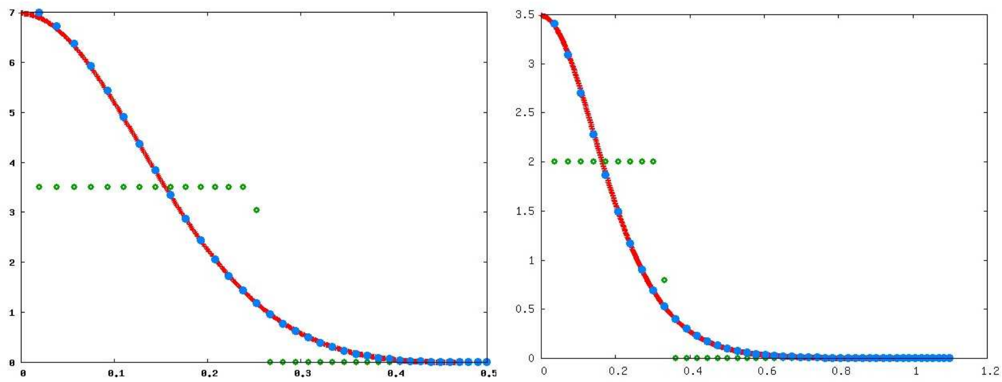

Figure 6.1: Test of numerical minimization for two 3D steady-states with polytropic indexes $k = 0.5$ and $k = 1.0$. The red lines show the density profiles calculated with the ODE method, blue points show density profiles calculated with the numerical minimization of the reduced energy-Casimir functional. Green points show the initial state used in minimization (step function).

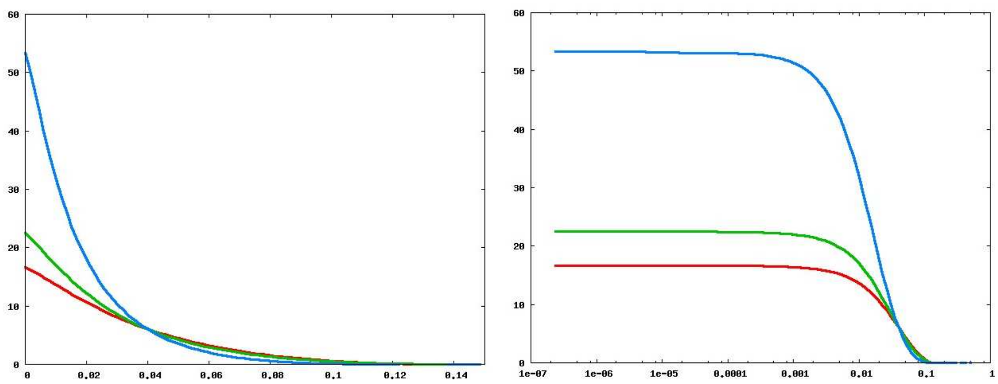

Figure 6.2: Numerical minimization of the reduced flat energy-Casimir functional for polytropes $k = 0.25, 0.5, 0.75$ (red,green and blue line) with normal and logarithmic horizontal scale.

numerical minimization provides only local minimizers of the reduced flat energy-Casimir functional and there is no guarantee that these minimizers are global as requested by the theory in Chapter 3 (even if the minimizers calculated in the full three-dimensional case are apparently global).

6.2 Approximation of flat solutions

This problem was already stated in [5], but we think it is important enough for us to mention it at least one more time. The real astronomical objects are of course not totally flat, but three dimensional with some finite (possibly small) thickness. This fact immediately leads to the following question: Is it possible to approximate the flat Vlasov-Poisson system by the full three dimensional Vlasov-Poisson system? In mathematical language, the problem stands as follows. We have some class of initial conditions

$$f_0^\varepsilon(x,v) := \tilde{f}_0(\tilde{x},\tilde{v})\delta^\varepsilon(x_3)\delta^\varepsilon(v_3).$$

Each of the functions above, when sufficiently regular with compact support, launches according to the standard existence theory a global classical solution of the Vlasov-Poisson system. According to the existence theory for the flat Vlasov-Poisson system (see [5]), function $\tilde{f}_0$ launches (again under some assumptions) a local classical solution of the flat Vlasov-Poisson system. Since f_0^ε approximates $\tilde{f}_0$ (in distributional sense), we would like to know if the solutions associated to f_0^ε approximate in any sense the solution launched from $\tilde{f}_0$.

6.3 3D-stability of flat objects

This problem is slightly connected to the previous one. All stability results for the flat Vlasov-Poisson system presented in this thesis prove stability against all perturbation living in a plane. This unphysical restriction is of course undesired since flat astronomical objects surely undergo perturbations perpendicularly to this plane. A very interesting (and according to our knowledge still open) question is, whether some stationary solution of the flat Vlasov-Poisson system keeps any of its stability properties even if we drop the above restriction for the perturbation.

The motivation which supports the conjecture, that some stability property should hold, lies in the analysis of the total energy functional for the three-dimensional approximation of some flat steady-state. We define this flat steady-state f_0 and its approximation f_0^ε as

$$\begin{aligned} f_0(x,v) &:= \tilde{f}_0(\tilde{x},\tilde{v})\delta_{x_3}\delta_{v_3}, \\ f_0^\varepsilon(x,v) &:= \tilde{f}_0(\tilde{x},\tilde{v})\delta^\varepsilon(x_3)\delta^\varepsilon(v_3), \end{aligned}$$

where δ^ε is the usual approximation of the Dirac's distribution. Now for the total energy we have the following estimate:

$$
\begin{aligned}
\mathcal{H}(f_0^\varepsilon) &= \frac{1}{2}\iint |v|^2 \tilde{f}_0(\tilde{x},\tilde{v})\delta^\varepsilon(x_3)\delta^\varepsilon(v_3)\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{2}\iint \frac{\rho_{\tilde{f}_0}(\tilde{x})\rho_{\tilde{f}_0}(\tilde{y})\delta^\varepsilon(x_3)\delta^\varepsilon(y_3)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \frac{1}{2}\iint |\tilde{v}|^2 \tilde{f}_0(\tilde{x},\tilde{v})\delta^\varepsilon(x_3)\delta^\varepsilon(v_3)\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{2}\iint \frac{\rho_{\tilde{f}_0}(\tilde{x})\rho_{\tilde{f}_0}(\tilde{y})\delta^\varepsilon(x_3)\delta^\varepsilon(y_3)}{|\tilde{x}-\tilde{y}|}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\iint |\tilde{v}|^2 \tilde{f}_0(\tilde{x},\tilde{v})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{v} - \frac{1}{2}\iint \frac{\rho_{\tilde{f}_0}(\tilde{x})\rho_{\tilde{f}_0}(\tilde{y})}{|\tilde{x}-\tilde{y}|}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} \\
&= \tilde{\mathcal{H}}(\tilde{f}_0).
\end{aligned}
$$

As we see, the energies of the approximations are always greater than the energy of the flat steady-state. We of course cannot build any result whatsoever on this calculation. The approximation we chose is very special and there is a chance that the energy inequality fails to hold if we take some different approximation. We also cannot expect that some flat steady state minimizes the total three-dimensional energy, because we already know how the global energy minimizers look like and they are spherically symmetric and not flat (see [15, 17, 12] etc.). What could be true is that the flat steady state is a local minimizer of the energy in some sense. However, the analysis of local minimizes of the energy functional lies momental beyond the reach of the energy-Casimir method.

6.4 Stability of planetary rings

In recent work [31], the energy-Casimir method was used to prove the existence and nonlinear stability of stationary shell solutions of the three-dimensional Vlasov-Poisson system with a fixed central point mass. This type of solutions can be used for example as a model for quasars, astronomical objects believed to be massive black holes surrounded by galactic matter. It is very probable that the same approach should work in the flat case as well. Now, if we replace the central point mass with some spherically symmetric stationary solution, which has the same mass, we get a reasonable model describing the dynamics of planetary rings. Note that the external potential acting on the flat shell would be the same, hence the existence and all properties of such stationary solutions would not alter from the ones with point mass in the center. A careful analysis of the variational problem under this setup would surely provide some stability results.

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